UNIT 8

Geometry

Introduction

Geometry is the study of angles, lines, shapes and solids. The word is derived from the Greek words ‘geo’ (earth) and ‘metria’ (measurement), and geometry has been used for thousands of years for measuring land, navigating, and mapping the world and the stars. It is now used in many different applications, from global positioning satellites to computer graphics and cosmology (the study of the physical universe as a whole).

A number of ancient Greek mathematicians laid the foundations of geometry. You may have heard of one of the foremost of them.

Much of the content of Euclid’s *Elements* was the core of school geometry in the UK until about 1970.

Euclid was a Greek mathematician who worked in the city of Alexandria in Egypt in the third century BC. Little is known about his life other than that he produced ten works, of which five have survived to the present day. His reputation as a mathematician is based on his main work, the *Elements*, which was the standard introduction to mathematics for over two thousand years. It is claimed to be the best-selling mathematics book of all time, and is one of the most frequently printed books ever.

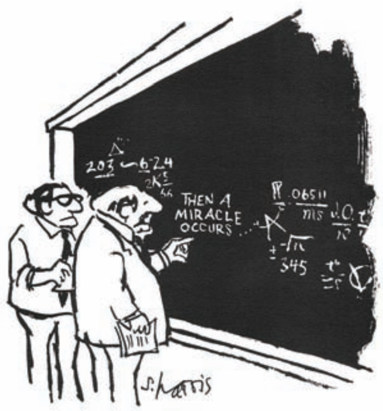
The following activity looks at Euclid’s *Elements*.

Video

Activity 1 *Learning about Euclid’s Elements*

Watch the video *Euclid’s Elements*.

One reason why the *Elements* was so important was that it introduced generations of mathematicians to ideas of rigorous proof. It starts with a small number of **axioms** (truths that were taken as self-evident), such as the fact that a straight line can be drawn between any two points. It then proceeds to prove theorems, such as Pythagoras’ Theorem (which you will meet in Section 3), using nothing more than the axioms and previously proven theorems. This approach was very influential in the development of mathematics.



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Although it isn’t possible to take such a formal approach here, you will see some examples of proving geometric theorems by using definitions and previously established results. This will give you a taste of the way that geometry was developed in Euclid’s *Elements*.

Section 1 introduces you to some geometric terminology and some important properties of angles between straight lines that you will be using throughout the rest of the unit. Section 2 concentrates on the properties of polygons – shapes with straight sides – and symmetry. In both Sections 1 and 2, you will be referred to some dynamic geometry software that allows you to explore geometric diagrams interactively, before looking at more rigorous proofs.

Section 3 considers when two shapes are essentially the same and introduces the ideas of congruency and similarity. These are useful both in practical applications such as constructing buildings and bridges, and in proving general properties of shapes.

The perimeters and areas of shapes are considered in Section 4. A modern application discussed here is the detection of abnormal cells in tissue samples by computer-assisted classification. You will also see Archimedes' method for calculating approximations for the constant π, which occurs in the formulas for the perimeter and area of a circle.

Finally, Section 5 looks at how to calculate the volume and the surface area of a solid object.

If you would like some extra practice with some of the ideas in this unit, then have a look at Maths Help Module 7: Geometry.

In 1706 the English mathematician William Jones introduced the use of π to mean the ratio of the circumference of a circle to its diameter, but the symbol didn't come into general use until it was popularised by Leonhard Euler in his *Introductio in analysin infinitorum* in 1748.

1 Angles

1.1 Angles and lines

This subsection introduces some terminology and notation that are useful for explaining geometric ideas clearly and concisely. You may like to make a note of any terms that you have not met before, so that you can refer to them easily as you work through the unit.

Let's start with some definitions. In geometry, a **point** has a position but no size. For example, the place where two lines cross (or two line segments meet) is a point. In Figure 1, six points are labelled with the upper-case letters A, B, C, D, E and F.

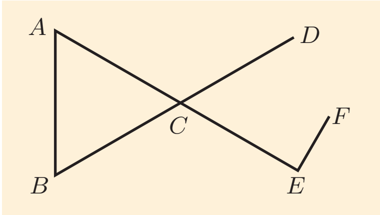


Figure 1 Points and line segments

A **line** is a straight line, as normally understood, but one that extends infinitely far in both directions. A finite portion of a line, which is all that you can draw in practice, is called a **line segment**. There are several line segments in Figure 1, such as the one between the points A and B, which is referred to as AB (or BA). Line segments are often just called lines, for brevity. A point where two line segments meet or cross is called a **vertex**. So in Figure 1 there are vertices at A, B, C and E.

In MU123 you may assume that any line segment drawn in a diagram is a straight line, even where a point along its length is identified. For example, in Figure 1, assume that ACE and BCD are both straight lines.

The plural of *vertex* is *vertices*.

Angles are a measure of rotation and can be measured in **degrees**. There are 360 degrees (written as 360°) in a full turn, and therefore there are 180° in a half-turn and 90° in a quarter-turn or **right angle**. In Figure 1, if the line segment EF is rotated through a quarter-turn anticlockwise, then it lies in the same direction as the line segment EC, so the angle between EF and EC is a right angle. This angle can be referred to as $\angle FEC$ or $F\widehat{E}C$ (or $\angle CEF$ or $C\widehat{E}F$).

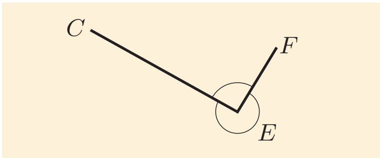


Figure 2 Two angles determined by the line segments FE and EC

In fact, there are *two* angles determined by the line segments FE and EC, one of 90° and one of 270°, as indicated by the small arcs in Figure 2. The notations $\angle FEC$ and $F\widehat{E}C$ refer to the smaller angle.

$\angle FEC$ and $F\widehat{E}C$ are both read as 'angle F E C'.

Finally, a **plane** is a flat surface that extends infinitely far in all directions. For example, a flat piece of paper is part of a plane.

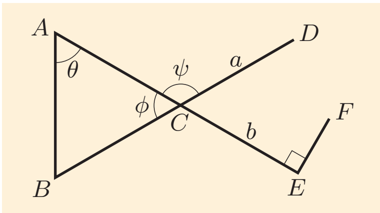


Figure 3 Line segments and angles labelled with single letters

Sometimes it is cumbersome to use several letters to refer to a line segment or angle, and in Figure 3 some line segments and angles have been labelled with single letters. The letters a and b have been used to label two line segments, and the Greek letters θ, ϕ and ψ have been used to label three angles, which are also marked with small arcs to help you to identify them. Also included in the diagram is the special square symbol that indicates a right angle. It has been used to mark $\angle CEF$ as a right angle.

Greek letters are often used to label angles. Some Greek letters that are used frequently are listed in Table 1, with their spellings and pronunciations. A full table of Greek letters is included in the Handbook.

Table 1 Some Greek letters

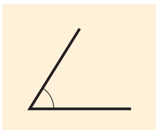
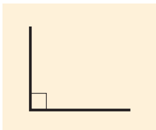
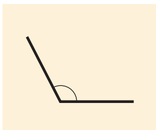
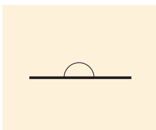
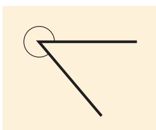
α	alpha	θ	theta (pronounced '*thee*-ta')
β	beta (pronounced '*bee*-ta')	ϕ	phi (pronounced '*fy*')
γ	gamma	ψ	psi (pronounced '*sigh*')
δ	delta	ω	omega (pronounced '*oh*-meh-ga')

The notation and symbols used for line segments and angles are also used to refer to the *lengths* of line segments and to the *sizes* of angles. So you can ask questions about Figure 3 such as: 'Is AB equal to BC?' or 'Is θ equal to $60°$?'

Angles can be classified into different types, as described in Table 2.

The mathematician and astronomer Hipparchus of Nicea (ca. 180–125 BC) is thought to have chosen 360 for the number of degrees in a full turn. Earlier, Babylonian astronomers had divided the day into 360 parts.

Table 2 Types of angle

Angle	Diagram	Description
Acute angle		Greater than 0° and less than 90°
Right angle		Equal to 90°
Obtuse angle		Greater than 90° and less than 180°
Straight angle		Equal to 180°
Reflex angle		Greater than 180° and less than 360°

Angles on a straight line

Since a straight angle is 180°, any angles that together make up a straight angle add up to 180°. For example, in Figure 4, $\angle ABC$ and $\angle CBD$ together add up to 180°. So, since $\angle ABC = 30°$,

$$\angle CBD = 180° - 30° = 150°.$$

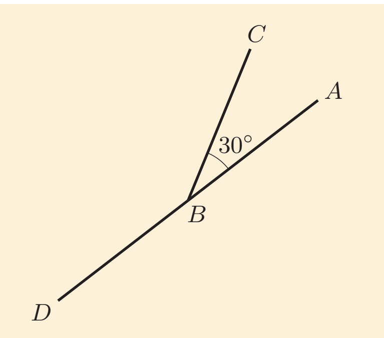


Figure 4 Two angles that add up to 180°

In Figure 5, the *three* angles α, β and γ add up to 180°; that is, $\alpha + \beta + \gamma = 180°$.

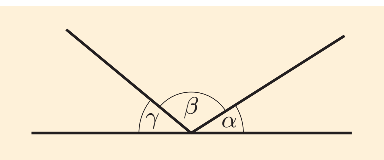


Figure 5 Three angles that add up to 180°

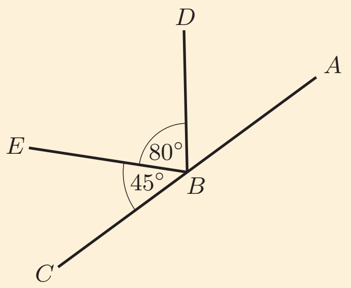
The general result is summarised below.

Angles on a straight line add up to 180°.

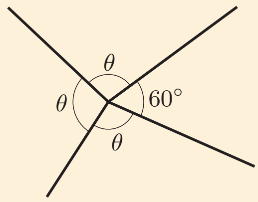
Many of the activities in this unit, and many of the applications of geometry, involve using a geometric diagram to deduce the sizes of angles or the lengths of line segments. When you do this, it is important to set out your solution clearly, justifying your results, as shown in the example below.

Example 1 *Calculating angles*

(a) Calculate $\angle ABD$ in the diagram below.



(b) Calculate the angle θ in the diagram below.



Solution

(a) State the facts that you are going to use.

ABC is a straight line, and angles on a straight line add up to 180°.

Write down an equation involving the unknown angle, and solve it.

So

$$45° + 80° + \angle ABD = 180°$$
$$\angle ABD = 180° - 80° - 45°$$
$$\angle ABD = 55°.$$

(b) Angles in a full turn add up to 360°. So

$$\theta + \theta + \theta + 60° = 360°$$
$$3\theta + 60° = 360°$$
$$3\theta = 300°$$
$$\theta = 100°.$$

You can write solutions to problems like those in Example 1 a little more concisely if you wish. For example, the solution to part (a) could be written with the justification in brackets like this:

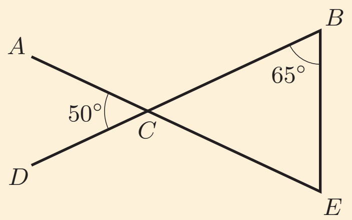
ABC is a straight line. So

$$\angle ABD = 180° - 80° - 45° \quad \text{(angles on a straight line)}$$
$$= 55°.$$

The following activities tie together some of the ideas that you've met in this subsection.

Activity 2 *Calculating angles*

This question is about the diagram below.



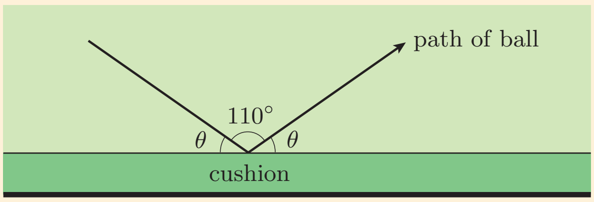
(a) Mark $\angle ACB$ as α and calculate its value.

(b) Which angles in the diagram are obtuse?

(c) Label the reflex angle at B as ω and calculate its value.

Activity 3 *Calculating angles of snooker balls*

When a snooker ball hits the cushion of a snooker table, the angle that its path makes with the cushion is the same after the impact as before. This angle is marked as θ in the diagram below. The diagram shows a path whose parts before and after impact make an angle of 110° with each other.



Calculate the angle θ in this case.

1.2 Pairs of equal angles

You have seen that you can sometimes deduce the sizes of angles from other angles that you already know. The next activity is about how you can do this in some particular situations. In the first part of the activity you will look at the angles formed when two lines cross each other, and in the second part you will consider the angles formed when a line crosses two parallel lines. As you saw in Unit 6, lines on a flat surface are *parallel* if they never cross even when extended infinitely far in each direction.

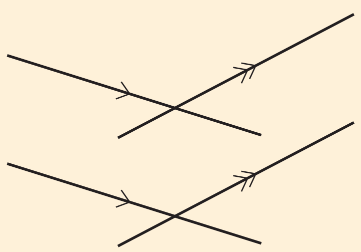


Figure 6 Two pairs of parallel lines

In a geometric diagram, parallel lines are indicated by putting matching arrowheads on the lines. When a diagram contains two pairs of parallel lines, one pair is marked with a single arrowhead on each line and the other pair is marked with a double arrowhead on each line. Figure 6 shows such a diagram.

When you are looking at a geometric diagram, you should take care not to assume properties that are not marked. For example, you should not assume that two lines that *look* parallel really *are* parallel, unless they are marked with matching arrowheads.

Activity 4 *Exploring angles*

Dynamic geometry

Open the dynamic geometry resource for Unit 8.

(a) Click on the 'Opposite angles' tab and follow the instructions in the left-hand panel.

(b) Click on the 'Alternate angles' tab and follow the instructions.

The rest of this subsection explains the results in Activity 4 in more detail, and shows you how they can be useful.

Opposite angles

Figure 7 shows two lines, AC and DB, which cross. The angles θ and ϕ in this figure are called a pair of **opposite angles** or, more informally, **X angles**. The angles ψ and ω form a second pair of opposite angles.

Some resources use the phrase 'vertically opposite' to describe opposite angles, since they occur at a vertex.

In Activity 4(a) you saw that opposite angles seem to be equal. However, looking at some examples, as you did in this activity, isn't sufficient to show that this result is always true. What is needed is a more formal proof, using the same sort of rigorous argument that Euclid used.

Activity 5 *Proving Euclid's proposition about opposite angles*

This activity leads you through the steps in Euclid's proof that opposite angles are equal.

Consider the opposite angles θ and ϕ in Figure 7.

(a) The angles θ and ψ lie on the straight line BD. Use this fact to find an equation relating θ and ψ.

(b) The angles ϕ and ψ lie on the straight line AC. Use this fact to find an equation relating ϕ and ψ.

(c) Use algebra to show that $\theta = \phi$, by eliminating ψ from the equations found in parts (a) and (b).

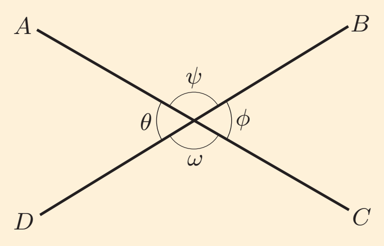


Figure 7 Two pairs of opposite angles

From Activity 5, we have the following result.

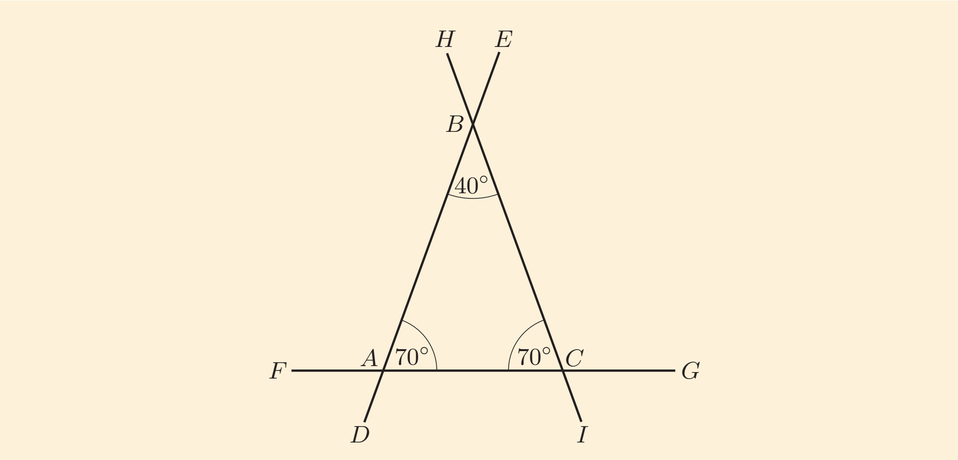
Opposite angles are equal.

This is a useful result, as it means that as soon as you spot a pair of angles that are opposite to each other, you can deduce that they are equal. For example, in Figure 7, $\psi = \omega$.

Try using this result, and other facts that you have learned so far, in the next activity.

Activity 6 *Finding opposite angles*

Look at the diagram below, where the sizes of the angles inside triangle ABC are given.



Find $\angle HBE$, $\angle FAD$ and $\angle BAF$.

Corresponding and alternate angles

Now let's look at the angles formed when a line crosses a pair of parallel lines, as shown in Figure 8.

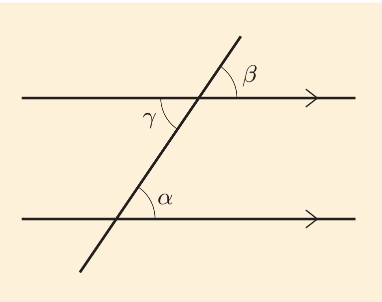


Figure 8 Corresponding and alternate angles

The angles α and β in Figure 8 are called **corresponding angles**, because they are in corresponding positions on the two parallel lines.

These angles are equal, because if you slide angle α up then it lies exactly on top of angle β, as you saw in Activity 4(b). Any corresponding angles can be seen to be equal in the same way, which gives the important result stated below.

> Corresponding angles are equal.

In Figure 9, two pairs of corresponding angles are marked on the same diagram (and there are two more pairs, which are not marked). One pair of corresponding angles is marked with single arcs, and a second pair is marked with double arcs. This is a convention used frequently in geometric diagrams when angles are not labelled with individual letters or with their sizes: equal angles are indicated by marking them with the same number of arcs.

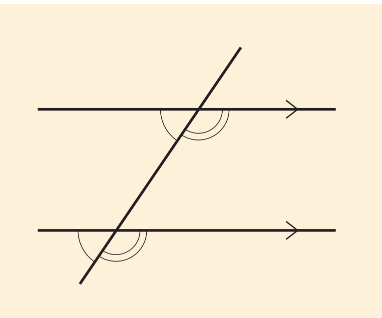


Figure 9 Two pairs of corresponding angles

The two angles marked with double arcs are part of a capital F, and so corresponding angles are also known informally as **F angles**.

Now look at Figure 8 again. The angles α and γ are known as **alternate angles**, because they are on alternate sides of the line that crosses the pair of parallel lines. They are also known informally as **Z angles**, because there is a pair of such angles in a capital Z.

You have seen that the angles α and β in Figure 8 are equal since they are corresponding angles. Also, the angles β and γ are equal since they are opposite angles, and hence the alternate angles α and γ are equal – you saw this argument in Activity 4. A similar argument applies to other pairs of alternate angles, so we have the following result.

Alternate angles are equal.

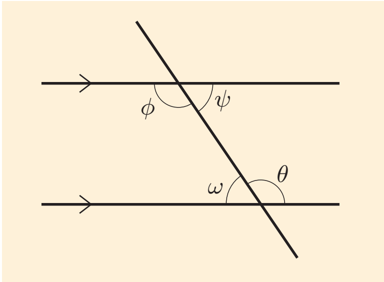


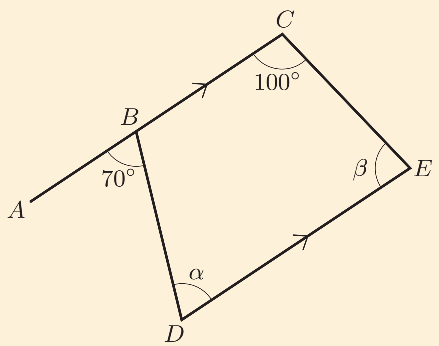
Figure 10 Two pairs of alternate angles

Not all pairs of alternate angles *look* like angles in a letter Z! For example, in Figure 10 the two angles marked θ and ϕ are obtuse angles, but they are alternate angles nevertheless. The angles marked ψ and ω are also alternate angles.

The next example illustrates how the results about angles that you have met in this subsection can be used to find unknown angles.

Example 2 *Finding corresponding and alternate angles*

Calculate the angles α and β in the diagram below.



Solution

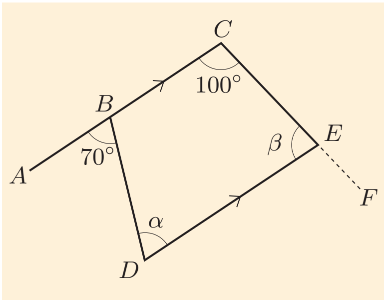
Look for alternate, corresponding and opposite angles.

The line segments AC and DE are parallel, so $\angle ABD$ and $\angle BDE$ are alternate angles.

So $\alpha = 70°$.

Add a line segment to the diagram to help you spot equal angles.

Extend CE to a point F, as shown in the margin. Then since AC and DE are parallel, $\angle DEF$ and $\angle BCE$ are corresponding angles. Since $\angle BCE = 100°$, it follows that $\angle DEF = 100°$.



Since β and $\angle DEF$ are angles on a straight line,

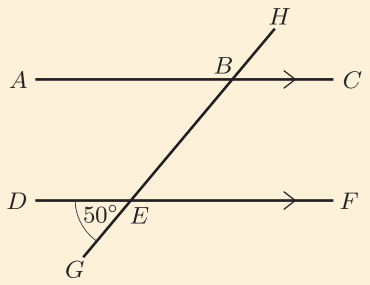
$$\beta = 180° - \angle DEF = 180° - 100° = 80°.$$

When you work on a problem like that in Example 2, you will probably find it helpful to mark the sizes of the angles on the diagram as you find them.

Here is an activity on equal angles.

Activity 7 *Finding angles equal to a given angle*

In the diagram below, $\angle DEG = 50°$ (this is marked on the diagram).



Which other angles in the diagram are equal to 50°, and why?

The result about alternate angles that you have met in this subsection can be stated as follows. Suppose that two lines are crossed by a third line, as shown in Figure 11. The result (in terms of the diagram in Figure 11) is:

If the first two lines are parallel, then the angles α and β are equal.

This result also works in reverse, in the sense that what is known as the *converse* result is true:

If the angles α and β are equal, then the first two lines are parallel.

In general, the **converse** of the result 'If A is true, then B is true' is the result 'If B is true, then A is true'.

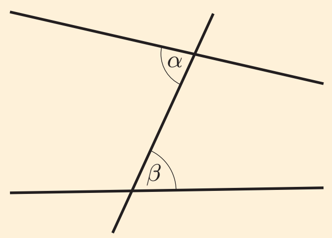
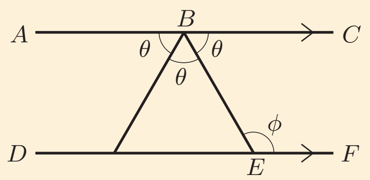


Figure 11 Two lines and a third line crossing them. Two angles are labelled α and β, respectively.

There is an important point here. A large proportion of mathematical results are of the form 'If A is true, then B is true' – and it is *not always the case* that the converse of a result is also a mathematical result. For example, if the last digit of a number is 2, then the number is even; this is a mathematical result. But the converse 'If a number is even, then its last digit is 2' is false, since (for instance) 14 is even.

Activity 8 *Finding an angle*

In the diagram below, the three angles marked θ are equal. Find the angle marked ϕ.



In this section you have seen how to use some results about angles to find the sizes of unknown angles. The same results can be used to prove general facts about shapes, as you will see in the next section.

2 Shapes and symmetry

2.1 Triangles

This subsection is all about triangles. You can refer to a triangle by using vertex labels. For example, the triangle in Figure 12 is referred to as triangle ABC. The notation $\triangle ABC$ is often used for brevity.

The **interior angles** of a triangle are the angles formed inside the triangle by its sides. For example, the interior angles of the triangle in Figure 12 are marked as α, β and γ. You may be familiar with the fact that the interior angles of every triangle add up to 180°. In the next activity you are asked to check this result for some triangles, and then prove it by using results that you found earlier.

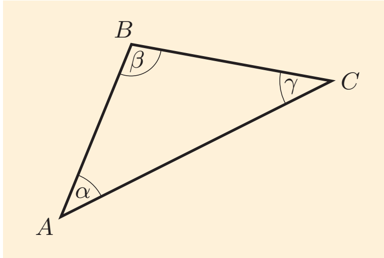


Figure 12 The interior angles of a triangle

Dynamic geometry

Activity 9 *Investigating the interior angles of a triangle*

Open the Unit 8 dynamic geometry resource, click on the 'Triangle' tab, and follow the instructions in the left-hand panel.

The result found in Activity 9 is as follows.

> The angles in a triangle add up to 180°.

The proof in Activity 9 is one of the most famous proofs in mathematics! It's worth thinking about the process of finding a proof like this in a bit more detail.

First, a key step was experimenting with the triangle, spotting a pattern and deciding what result seems to be true in general. Then, using results that were already proven, and working step by step with a clear diagram, it was possible to prove the result.

The idea of adding a line to the triangle is one that may not have occurred to you, although you have had a small taste of this in Example 2. An addition to a geometric diagram, in order to help prove a fact about the original shape, is known as a **construction**. A construction that is a line is known as a **construction line**. The extra line added to the diagram in Activity 9 is an example of a construction line. Adding construction lines to a geometric diagram can be a very powerful technique.

Once you have come up with the main ideas of a proof, the next stage is to write the proof out clearly, using words and mathematical notation, so that a reader can understand it. You must give an argument that refers to results known to be true, and work logically step by step from what you know to what you want to prove. The process of setting out a geometric argument is summarised overleaf, and then Example 3 shows how it is applied to prove that the angles in a triangle add up to 180°.

> ***Setting out a geometric argument***
>
> 1. State the general fact that is to be proved, including any given information.
> 2. Draw a diagram that contains the information that is given, labelling the important features, such as points and angles.
> 3. Add any useful constructions.
> 4. Proceed step by step from what is given to what is to be proved, explaining your reasoning clearly.

Example 3 *Proving the result about the angles in a triangle*

Prove that the angles of any triangle add up to 180°.

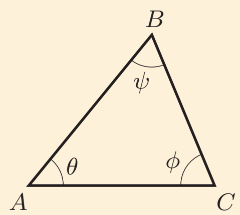
Solution

State the general fact to be proved.

Here we prove that the angles in a triangle add up to 180°.

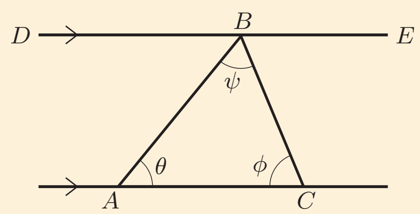
Draw a suitably-labelled diagram containing the information that you know.

Consider $\triangle ABC$.



Add any necessary constructions.

Draw a line through B, parallel to the side AC.



Proceed step by step from what you know to the fact that is to be proved. Explain each step clearly.

$\angle DBA$ and $\angle BAC$ are alternate angles. Therefore $\angle DBA = \theta$.

$\angle EBC$ and $\angle BCA$ are alternate angles. Therefore $\angle EBC = \phi$.

$\angle DBA$, $\angle ABC$ and $\angle EBC$ are on the straight line passing through B and so add up to 180°.

So $\theta + \psi + \phi = 180°$.

But this is the sum of the angles in $\triangle ABC$.

Thus the angles in a triangle add up to 180°.

You will be looking at some more proofs later in the unit, but in the rest of this subsection you will see how the result about the sum of the angles in a triangle can be used in different situations. First we look at two special types of triangle: *equilateral* and *isosceles* triangles.

'Isosceles' is pronounced 'eye-*sos*-eh-lees'.

If a triangle has all its sides the same length, then all its angles are equal and the triangle is known as an **equilateral triangle**. On a geometric diagram, you can show that two or more line segments have the same length by putting a stroke, or the same number of strokes, on each of the line segments, as shown in Figure 13.

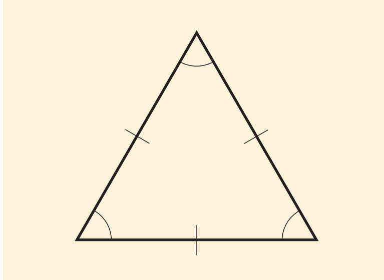


Figure 13 An equilateral triangle

Activity 10 *The angles of an equilateral triangle*

Show that each angle in an equilateral triangle is 60°.

The result that you were asked to show in Activity 10 is worth remembering, so it is stated formally below.

Each angle of an equilateral triangle is 60°.

If a triangle has just *two* sides that are the same length, then it also has two equal angles, known as the **base angles**, and the triangle is called an **isosceles triangle**. As shown in Figure 14, the third angle is known as the **apex angle**.

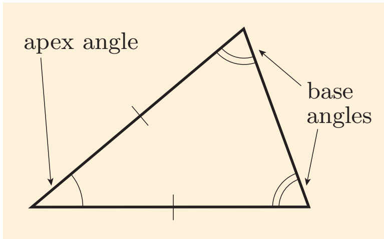
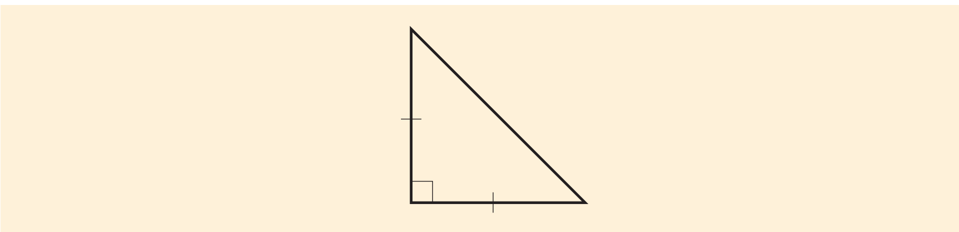


Figure 14 An isosceles triangle

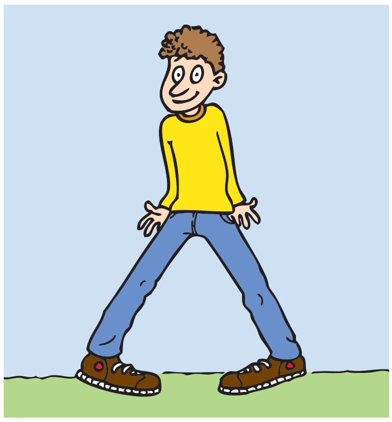
A triangle in which one angle is equal to 90° is called a **right-angled triangle**. Example 4 considers the angles in a right-angled triangle that is also isosceles.

Example 4 *Finding angles in an isosceles triangle*

Calculate the base angles in an isosceles right-angled triangle, such as the one shown below.



The word 'isosceles' comes from the Greek *isos* (same) and *skelos* (leg).



Solution

Start from what you know.

Let each base angle of the triangle be θ (the same letter can be used for each angle since the angles are equal). Then the angles of the triangle are 90°, θ and θ.

Using the fact that the angle sum of a triangle is 180° gives the equation

$$90° + 2\theta = 180°.$$

Solve the equation.

Rearranging the equation gives

$$2\theta = 180^\circ - 90^\circ$$
$$2\theta = 90^\circ$$
$$\theta = 45^\circ.$$

So each base angle of the triangle is 45°.

Now try the following activity.

Activity 11 *Finding angles in isosceles triangles*

(a) Each base angle of an isosceles triangle is 62°. What is the apex angle?

(b) The apex angle of an isosceles triangle is 30°. What are the base angles?

If you can show that two angles in a triangle are equal to each other, then you can deduce that the triangle is isosceles (or possibly equilateral) and that the sides opposite these two angles have the same length.

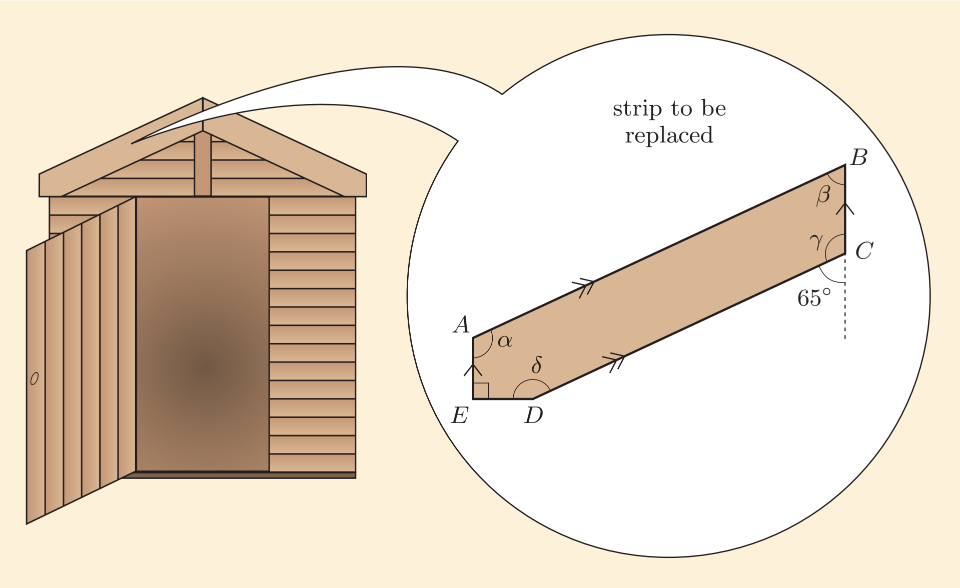
A triangle that is neither equilateral nor isosceles is known as a **scalene triangle**. All its sides are of different lengths.

The next example involves putting together several results that you have met so far. You will see that the steps of drawing a diagram, adding construction lines and working logically apply here too. The example involves finding some unknown angles: each new angle is found using information that was either known at the start or worked out earlier in the solution.

Tutorial clip

Example 5 *Finding angles related to a garden shed*

The diagram below shows the front of a garden shed, including a wooden strip that needs to be replaced. A close-up of the replacement strip is shown in the inset. The strip makes an angle of 65° with the vertical. The lines AB and DC are parallel, and the lines AE and BC (being vertical) are also parallel. Calculate the angles α, β, γ and δ.



Solution

Use the angle properties of straight lines, parallel lines and angles.

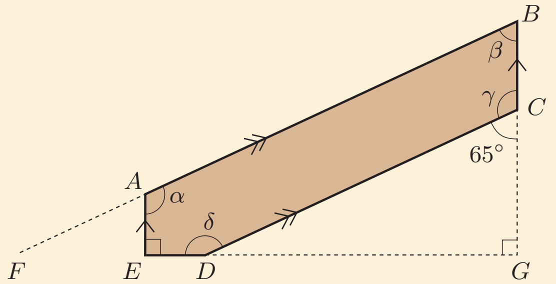
Since the angles on a straight line add up to 180°,

$$\gamma = 180° - 65° = 115°.$$

Since the lines DC and AB are parallel, the angles marked β and 65° are corresponding angles. So

$$\beta = 65°.$$

Draw construction lines to help you to find the unknown angles.



Since the lines AE and BC are parallel, $\angle FAE$ and $\angle ABC$ are corresponding angles. So $\angle FAE = \beta = 65°$.

Since the angles on a straight line add up to 180°,

$$\alpha = 180° - 65° = 115°.$$

Since the angles in $\triangle CGD$ add up to 180°,

$$\angle CDG = 180° - 90° - 65° = 25°.$$

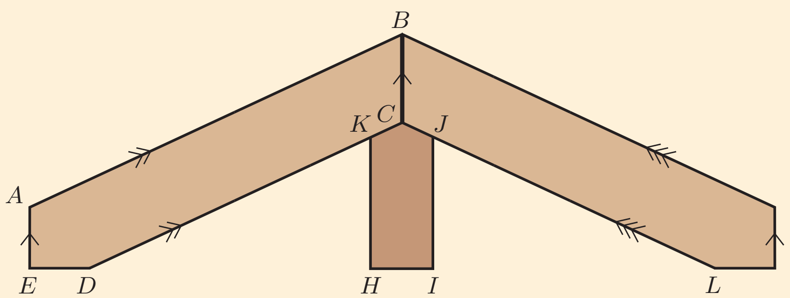
Hence, since the angles on a straight line add up to 180°,

$$\delta = 180° - 25° = 155°.$$

The following activity asks you to go through a similar process to find some unknown angles in another diagram. There are often many different ways of finding unknown angles, and you may find a different way from that in the given solution.

Activity 12 *More angles related to the shed*

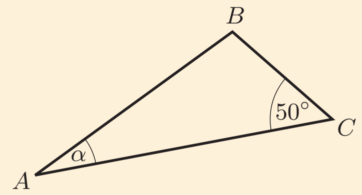
The wooden strip on the shed in Example 5 is one of a pair, as shown in the diagram below. Each of the two strips is at an angle of 65° to the vertical, and where they join there is a vertical strip, labelled as $KCJIH$ in the diagram below. This diagram extends the labelling notation that was used in Example 5. Find $\angle HKC$, $\angle KCJ$ and $\angle CJI$.



In Example 5 and Activity 12, the unknown angles were found by using known angles and working systematically around the diagrams. If no angles, or not enough angles, in a diagram are known, then it is sometimes useful to use letters to label one or more of the angles, and then find *expressions* for the other angles in terms of those letters.

Example 6 *Finding an expression for an angle*

Find an expression for $\angle ABC$ in the triangle below in terms of α.



Solution

Since the sum of the angles in a triangle is 180°,

$$\angle ABC + 50° + \alpha = 180°.$$

Hence

$$\angle ABC = 180° - 50° - \alpha$$

so

$$\angle ABC = 130° - \alpha.$$

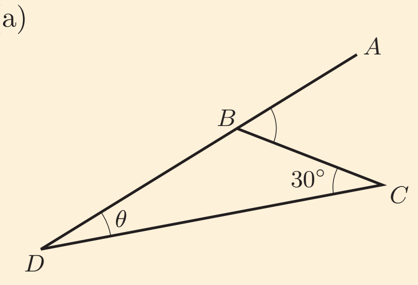
$\angle ABC$ can now be labelled as $130° - \alpha$ on the diagram.

Here is a similar activity for you to try.

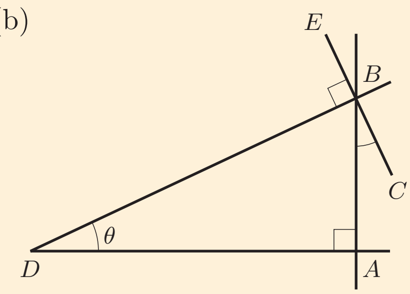
Activity 13 *Finding angles in terms of a variable angle*

For each of the following diagrams, find $\angle ABC$ in terms of θ.

(a)



(b)



Next, you are asked to use the dynamic geometry resource to discover something about the sum of the *exterior angles* of a triangle. An **exterior angle** of a triangle is the angle formed outside the triangle by one side and an extension of the **adjacent** side, as illustrated in Figure 15.

Activity 14 *Investigating the exterior angles of a triangle*

Dynamic geometry

Open the Unit 8 dynamic geometry resource and click on the 'Exterior angles' tab. Follow the instructions in the left-hand panel, and make a conjecture about the sum of the exterior angles in any triangle.

Prove your conjecture.

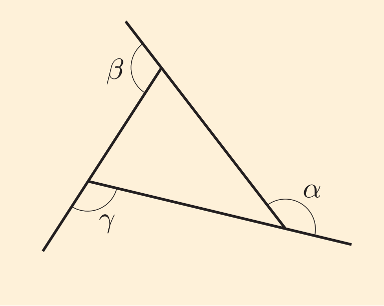


Figure 15 Exterior angles of a triangle

2.2 Polygons

A triangle has just three sides, but there are plenty of geometric shapes with more than three. A **polygon** is a plane shape with straight sides – so triangles and squares are examples of polygons. In particular:

- A **quadrilateral** is a polygon with four sides.
- A **pentagon** is a polygon with five sides.
- A **hexagon** is a polygon with six sides.

Similarly, **heptagons**, **octagons**, **nonagons** and **decagons** are polygons with, respectively, 7, 8, 9 and 10 sides. Figure 16 shows a few polygons.

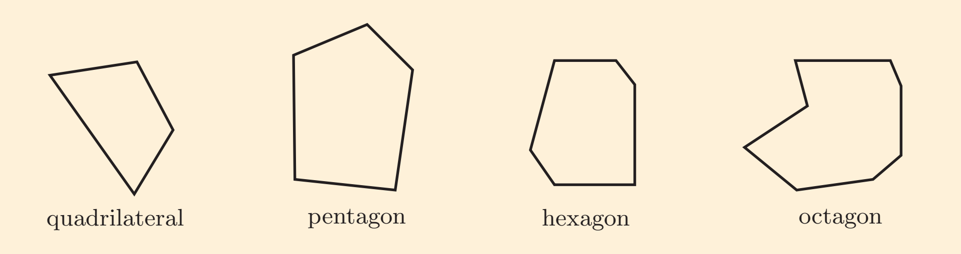


Figure 16 Some polygons

You have already seen some names that are used to describe different types of triangles: *equilateral triangle*, *isosceles triangle* and *scalene triangle*. Table 3 (overleaf) shows examples of some types of quadrilaterals and their associated properties.

In this table, and in places in the rest of the unit, **opposite angles** means the *internally opposite* angles of a quadrilateral. These are the angles not at the ends of the same side: in the quadrilateral $ABCD$ in Figure 17, the angles at A and C are internally opposite, as are those at B and D. (Note that this is a different concept of 'opposite' from the opposite angles in Section 1, which share a vertex. It is usually clear from the context which type of opposite angles is meant.)

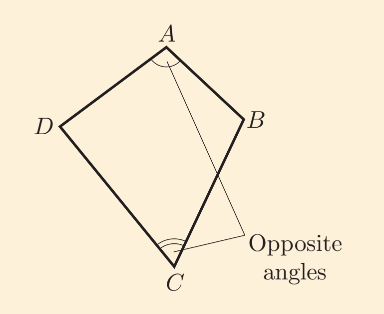
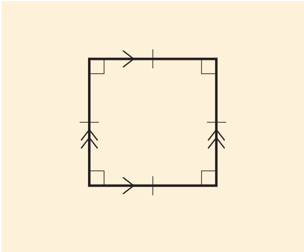
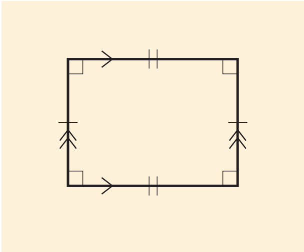
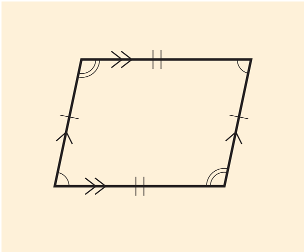
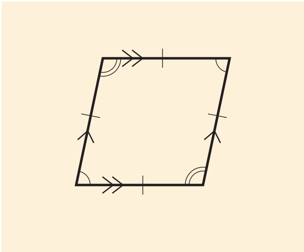
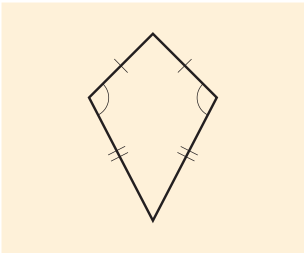
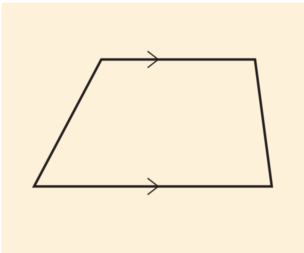


Figure 17 A pair of opposite angles in a quadrilateral

Table 3 Some special types of quadrilaterals

Shape	Diagram	Description
Square		Four equal sides and four right angles; opposite sides are parallel
Rectangle		Four right angles; opposite sides are equal and parallel
Parallelogram		Opposite sides are equal and parallel; opposite angles are equal
Rhombus		Four equal sides; opposite sides are parallel; opposite angles are equal
Kite		Two pairs of adjacent equal sides; one pair of opposite equal angles
Trapezium		One pair of parallel opposite sides

From Table 3 you can see that squares, rectangles and rhombuses are all special types of parallelogram.

Not all the properties in Table 3 are necessary to *define* the different types of quadrilateral. For example, a parallelogram can be defined to be a quadrilateral whose opposite sides are parallel, and the additional properties that opposite sides are equal and opposite angles are equal can be proved using this definition. In the next activity, you are asked to use the dynamic geometry resource to see how the property that opposite angles are equal can be proved. Later in the unit you will see how, if a kite is defined as a quadrilateral with two pairs of adjacent equal sides, then it can be proved that a kite also has one pair of opposite equal angles.

The properties of types of triangles, such as the property that an isosceles triangle has two equal angles, can be deduced from their definitions, in a similar way.

Activity 15 *Investigating the angles of a parallelogram*

Dynamic geometry

Open the Unit 8 dynamic geometry resource and follow the instructions in the 'Parallelogram' tab to see how to prove that opposite angles in a parallelogram are equal.

You saw earlier that the interior angles of any triangle add up to 180°. The sum of the interior angles of a polygon with more than three sides can be found by dividing the interior of the polygon into triangles.

For example, Figure 18 shows a pentagon divided into three triangles. The angle sum of each of the three triangles ($\triangle BCD$, $\triangle BDE$ and $\triangle ABE$) is 180°. But each angle in each triangle is an angle in the pentagon or a part of one of these angles. For example, $\angle ABC$ is divided into $\angle ABE$, $\angle EBD$ and $\angle DBC$. Moreover, each angle or part-angle in the pentagon belongs to one of the three triangles. So the angle sum of the pentagon is equal to the total angle sum of the three triangles, which is $3 \times 180° = 540°$.

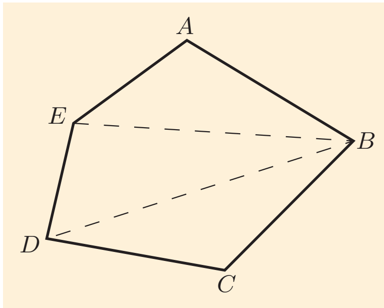


Figure 18 A pentagon divided into three triangles

Activity 16 *Calculating the angle sum of a hexagon*

Calculate the angle sum of a hexagon by dividing its interior into triangles.

A polygon is said to be **regular** if its sides are of equal length and its interior angles are equal. Some regular polygons are shown in Figure 19. (The polygons in Figure 16 on page 23 are not regular.)

Regular quadrilaterals are called squares, and regular triangles are called equilateral triangles.

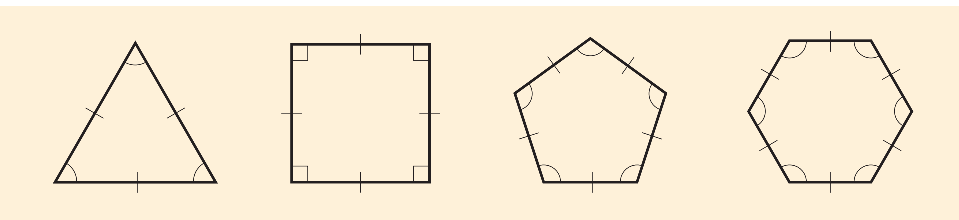


Figure 19 Some regular polygons

You can calculate the size of each angle of a regular polygon by dividing the total angle sum by the number of angles. For example, each angle in a regular pentagon is

$$\frac{540°}{5} = 108°.$$

Activity 17 *Finding the interior angles of a regular hexagon*

Calculate the interior angles of a regular hexagon.

Many geometric properties of shapes have practical applications. For example, in Mozambique farmers use geometric properties of rectangles to mark out the bases of their houses.

They use two equal lengths of rope tied together at their midpoints, and a piece of bamboo that is the width of the house that they plan to build. Two of the rope ends are attached to the ends of the bamboo, and the ropes are then pulled taut, as shown in Figure 20, making sure that each rope stays in a straight line. The endpoints of the ropes indicate the four corners of the house.

Source: Gerdes, P. (1988) 'On culture, geometrical thinking and mathematics education', *Educational Studies in Mathematics*, vol. 19, no. 2, pp. 137–62.

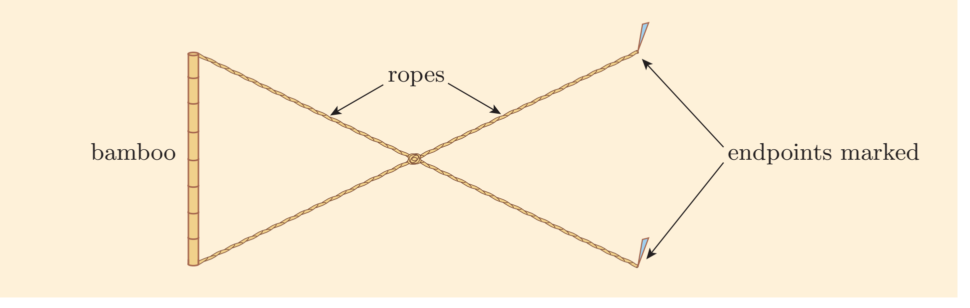
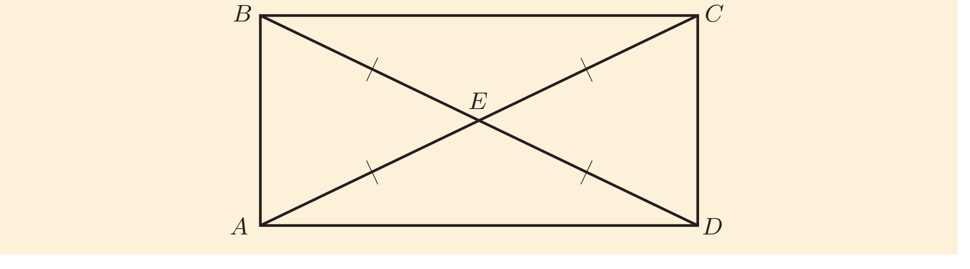


Figure 20 A device used in Mozambique for marking out a rectangle

The next activity asks you to explain why this method always gives a rectangle.

Activity 18 *Marking a rectangle for the base of a house*

In the diagram below, AB represents the piece of bamboo and BD and AC represent the ropes.



(a) Explain why $\angle ABE = \angle BAE$ and $\angle EBC = \angle ECB$.

(b) Label $\angle ABE$ as α. Find expressions, in terms of α, for the following angles.

(i) $\angle BAE$ (ii) $\angle AEB$ (iii) $\angle BEC$ (iv) $\angle EBC$

(c) By adding together the expressions for $\angle ABE$ and $\angle EBC$, show that $\angle ABC$ is a right angle.

(d) By a similar argument, the other interior angles of $ABCD$ can also be shown to be right angles. What can you deduce about the shape $ABCD$?

Remember that the only properties that you can assume are the four equal line segments marked, and the facts that AEC and BED are straight lines. For example, you can't assume that $\angle ABC$ is a right angle just because it looks like one. The fact that $\angle ABC$ is a right angle is the property that you're trying to *prove*! Similarly, you cannot assume that BC is parallel to AD.

The dynamic geometry resource includes an optional activity, 'Semicircle', which you may like to look at if you would like further practice in proving geometric results. This activity explores the size of the angle $\angle BAC$ in Figure 21, where A is any point on the curved part of the semicircle.

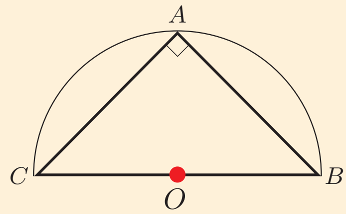


Figure 21 The angle in a semicircle

2.3 Symmetry

Rotational symmetry

Some shapes have the property that if you rotate them through a fixed angle (less than a full turn) about a fixed point, then the rotated shape looks the same as the original shape. Such a shape is said to have **rotational symmetry**, and the fixed point is called the **centre of rotation**.

For example, if an equilateral triangle is rotated through one third of a full turn (120°) about its centre, then the rotated triangle looks the same as the original triangle, as shown in Figure 22. Since there are three positions in which the rotated triangle looks the same, it is said to have rotational symmetry of **order** 3, or three-fold **rotational symmetry**. Another way to think about the three-fold rotational symmetry of the triangle is that three rotations are needed to return it to its starting position.

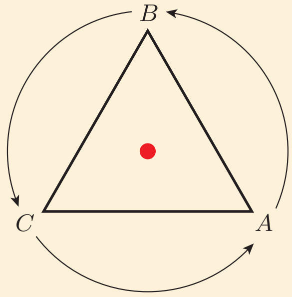


Figure 22 Rotating this equilateral triangle by 120° anticlockwise about its centre moves vertex A to vertex B, vertex B to vertex C, and vertex C to vertex A

All regular polygons have rotational symmetry. The order of the rotational symmetry is the same as the number of sides.

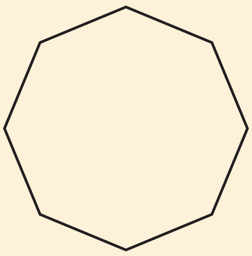
You can see many examples of rotational symmetry in nature. For example, a hibiscus flower (Figure 23) illustrates five-fold rotational symmetry, as you saw in the video for Unit 1.

Figure 23 A hibiscus flower

Activity 19 *Spotting rotational symmetries*

For each of the following pictures, state the order of rotational symmetry.

(a)



(b)

(c)

(d)

Line symmetry

Some shapes exhibit a different kind of symmetry, known as **line symmetry**, *mirror symmetry* or *reflectional symmetry*. For example, look at the first butterfly in Figure 24. If you were to fold the paper along an imaginary line down the centre of the stalk that the butterfly is resting on, then the pattern on one half of the butterfly would lie exactly on top of the pattern on the other half. Alternatively, if you place a mirror along this line, then the reflection gives the other half of the butterfly. This imaginary line is known as a **line of symmetry** (or a **reflection line** or *mirror line*). The second butterfly in Figure 24 also has a line of symmetry.

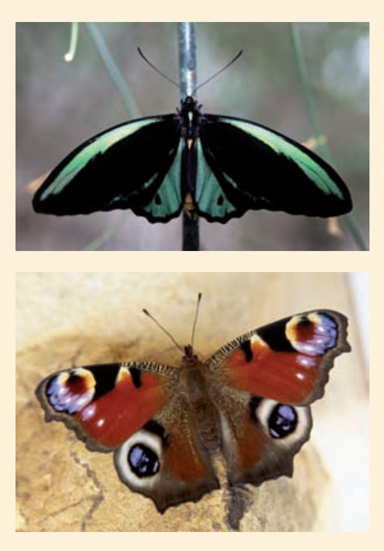


Figure 24 Symmetry in the wings of butterflies

Shapes can have more than one line of symmetry. For example, a rhombus has two lines of symmetry, as shown in Figure 25.

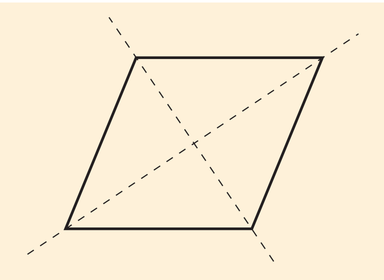


Figure 25 The two lines of symmetry of a rhombus

Activity 20 *Symmetries of quadrilaterals*

Draw an example of each of the following quadrilaterals and mark in any lines of symmetry. (You may find it useful to refer back to Table 3 on page 24 for the main properties of these shapes.)

(a) A square

(b) A rectangle (that is not a square)

(c) A parallelogram (that is neither a rectangle nor a rhombus)

(d) A kite (that is not a rhombus)

In Units 10 and 12 you will see how line symmetry can be a useful way of describing some graphs.

In this section you have seen how to set out a geometric argument formally and explored some properties of polygons, as well as looking at some practical applications. The next section extends these ideas by looking at some more properties of triangles and how they can be used, both practically and theoretically.

3 Congruent and similar shapes

Geometric figures with the same size and shape are said to be **congruent**. So two shapes are congruent if you can pick one of the shapes up and place it exactly on top of the other shape, rotating or flipping over the first shape if necessary.

You saw a pair of congruent shapes in Activity 12, which was about wooden strips on a garden shed. The shed is symmetrical, so the strip on the left-hand side is congruent to the strip on the right-hand side, as shown in Figure 26. Figure 27 shows some other pairs of congruent shapes.

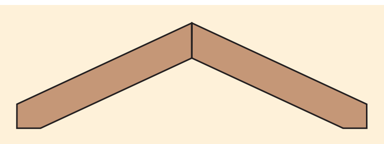


Figure 26 The wooden strips on the shed

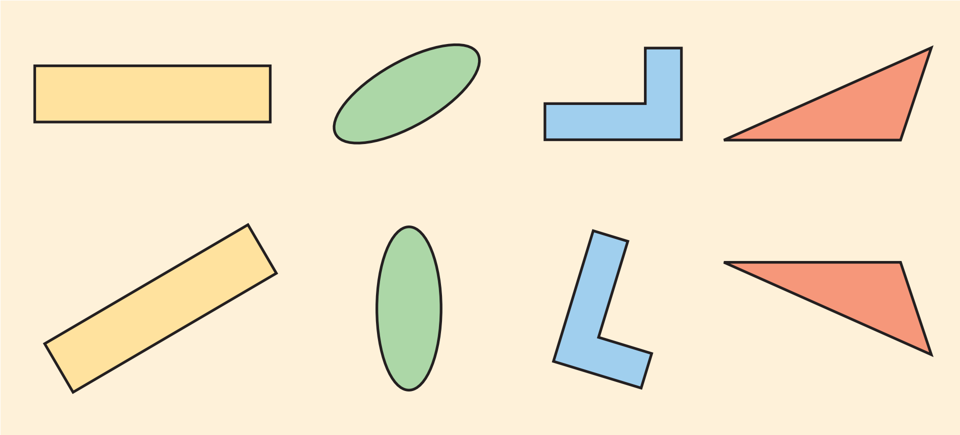


Figure 27 Four pairs of congruent shapes

Making congruent shapes, and checking that two shapes are congruent, occurs frequently in building and manufacturing. Since measuring sides and angles takes time, an important question to ask is: 'How many measurements are needed to make sure that two shapes are congruent?' Subsection 3.1 explores this question for *triangles*, and it uses some conditions for congruency of triangles to prove some geometric results. Triangles, and shapes made from triangles, are often used in constructions such as buildings and bridges, both to make the structures rigid and for an attractive design – examples are shown in Figures 28 and 29.

Figure 28 The Bank of China in Hong Kong

Geometric figures that have the same shape, but not necessarily the same size, are said to be **similar**. For example, if one shape is an enlargement of another (again, flipping is allowed), then the two shapes are similar, as illustrated in Figure 30.

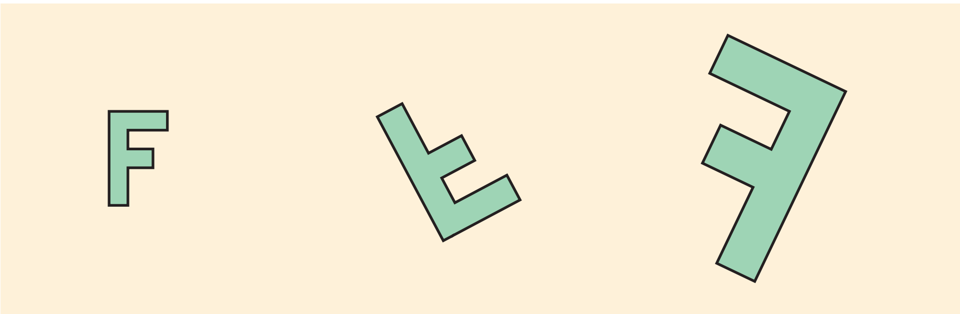


Figure 30 Three similar shapes

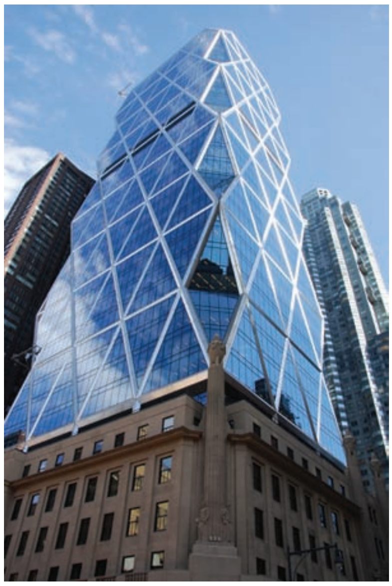


Figure 29 The Hearst Tower in New York

As you saw in Unit 3, the factor by which the lengths in one shape are enlarged with respect to the lengths in a similar shape is called a **scale factor**. For example, each length in the second (flipped) 'F' shape in Figure 30 is about 1.5 times the corresponding length in the first 'F' shape, so the scale factor from the first shape to the second is about 1.5.

Subsection 3.2 investigates when two triangles are similar, and illustrates how similar triangles are used in practice. Then in Subsection 3.3 you will see how similar triangles can be used to prove one of the most well-known theorems in geometry – *Pythagoras' Theorem*.

3.1 Congruent triangles

Figure 31 shows three congruent triangles. Each of the first two triangles has its sides marked with one stroke, two strokes and three strokes in clockwise order, so either one of these triangles can be placed on top of the other without flipping. The sides of the third triangle have the same marks, but in anticlockwise order, so this triangle has to be flipped over before it can be placed on top of either of the other two.

Recall that strokes are used to indicate which side lengths are equal, and small arcs are used to indicate which angles are equal.

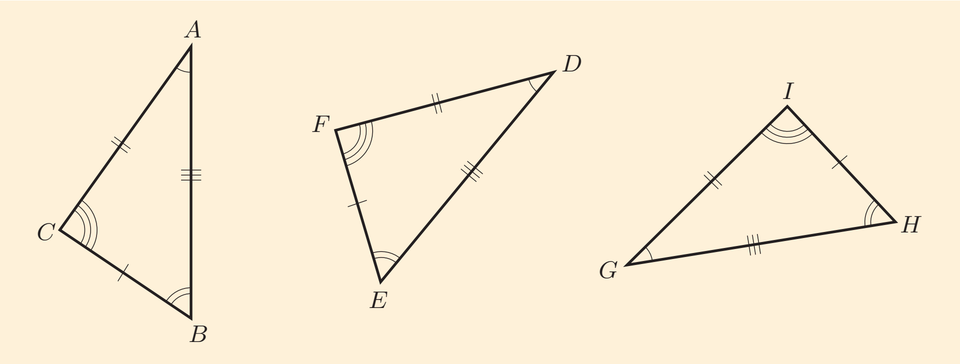


Figure 31 Three congruent triangles

When all the triangles are placed exactly on top of each other, the vertices A, D and G coincide, as do B, E and H, and also C, F and I. Vertices that can be made to coincide like this are said to be **corresponding vertices**. Since the angles marked with a single arc (at A, D and G) lie on top of each other, they are said to be **corresponding angles**. The angles marked with a double arc (at B, E and H) also form a set of corresponding angles, as do those marked with a triple arc (at C, F and I). (Note that this is a different use of the phrase 'corresponding angles' to earlier in the unit, where the same phrase referred to the F angles formed when a line crossed two parallel lines.)

The symbol $\cong$ is read as 'is congruent to'.

The symbol $\cong$ is used to indicate that two shapes are congruent. So you can write

$$\triangle ABC \cong \triangle DEF \cong \triangle GHI \tag{1}$$

to indicate that the three triangles in Figure 31 are congruent. In a statement like this, the order of the vertices indicates which vertices correspond. So statement (1) tells you that A, D and G correspond, as do B, E and H, and finally C, F and I.

Conditions for congruency

It might seem that to show that two triangles are congruent, you would need to gather six pieces of information for each triangle: three lengths and three angles. But how much of this information is actually needed to determine whether two triangles are congruent?

To work this out, let's consider how much information about a triangle you need to have in order for the triangle to be completely determined. In other words, let's look at how much information is needed to ensure that there is only one possible shape and size for the triangle.

The six possible pieces of information are the lengths of the three sides and the sizes of the three angles. Certainly it is not enough to know only one or two of these pieces of information. If you know the lengths of two sides of a triangle, or the sizes of two angles, or the length of one side and the size of one angle, then there is more than one possibility for the triangle.

For example, Figure 32 shows two different triangles, each of which has sides of lengths 2 cm and 3 cm.

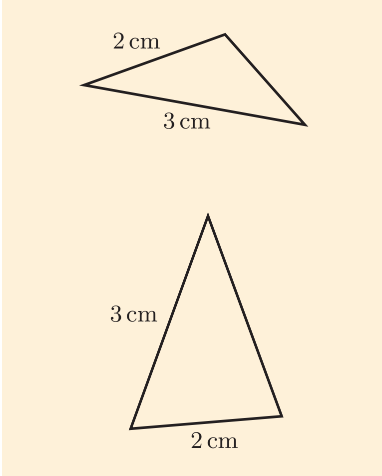


Figure 32 The lengths of two sides of a triangle do not constitute enough information to determine the triangle

So you certainly need at least three of the six possible pieces of information for the triangle to be determined. Let's consider the possible cases where you know three pieces of information.

You could know all three sides, all three angles, two sides and an angle or two angles and a side. The first of these cases is known as the *side-side-side* case (**SSS**), and the second is known as the *angle-angle-angle* case (**AAA**). Each of the third and fourth cases splits into two subcases, as follows.

If you know two sides and an angle, then the angle could be between the two sides – this is the *side-angle-side* case (**SAS**) – or it could be one of the other two angles – this is the *angle-side-side* case (**ASS**). Similarly, if you know two angles and a side, then the side could be between the two angles – this is the *angle-side-angle* case (**ASA**) – or it could be one of the other two sides – this is the *angle-angle-side* case (**AAS**).

Let's consider these six cases in turn.

The side-side-side case (SSS)

First, let's consider the case where the lengths of the sides of a triangle are known. Does this completely determine the triangle?

You might like to experiment with three lengths of paper. Try to arrange them into a triangle in any way possible, as shown in Figure 33. You will find that however much you try, there is essentially only one way to make a triangle. So if the side lengths of a triangle are known, then there is only one possibility for the shape and size of the triangle. (You will see how to calculate the angles from the lengths of the sides in Unit 12.)

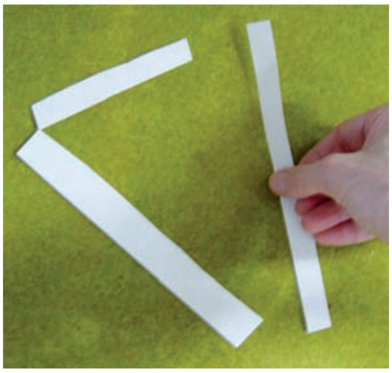


Figure 33 Trying to form a triangle

This means that if you know that the lengths of the sides in one triangle are the same as the lengths of the sides in another triangle, then the two triangles are congruent, as shown in Figure 34: $\triangle ABC \cong \triangle DFE$.

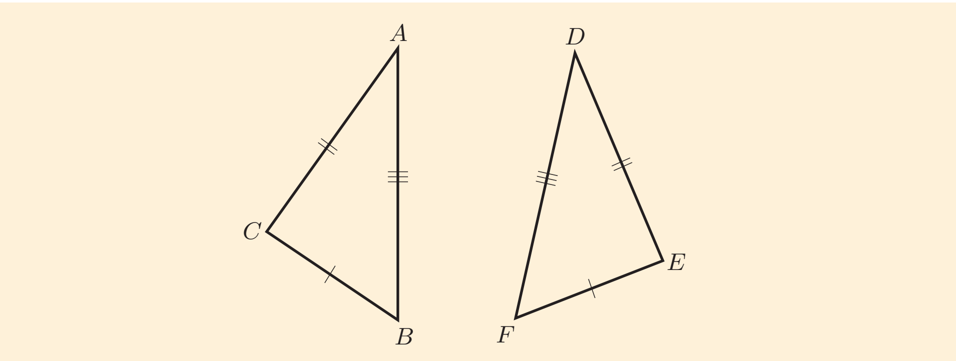


Figure 34 Two triangles that are congruent by SSS

Geometry started by appealing to physical intuition in situations such as this, but Euclid's approach in the *Elements* gave a method in which 'intuitively obvious' facts could be shown to be a logical consequence of more basic facts. This was a great step forward in mathematics.

Side-side-side congruency of triangles is important in structural engineering. If you join together three rods to form a triangle, then the structure is rigid, even if the joints are hinges. Compare this with four rods joined to make a square – this structure changes shape easily. Many engineering lattice structures are made up of triangles.

The side-angle-side case (SAS)

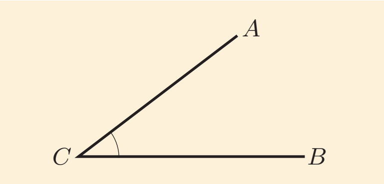


Figure 35 Two sides and an included angle

Consider the situation where you know the lengths of two sides of a triangle, together with the angle between them, which is called the **included angle**. This is illustrated in Figure 35. There is only one way to complete the triangle here, which is to draw the line from A to B. So the triangle is determined from this information.

So if two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, then the two triangles are congruent, as illustrated in Figure 36.

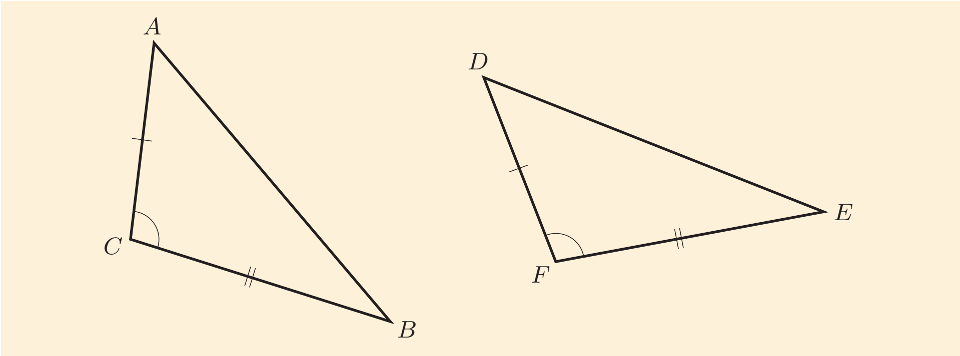


Figure 36 Two triangles that are congruent by SAS

The angle-side-angle case (ASA)

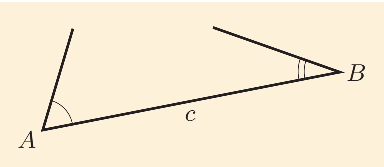


Figure 37 Two angles and the side between them

Suppose that you know two angles of a triangle and the side between them – the **included side** – as illustrated in Figure 37.

The point where the line segments from vertices A and B meet must be the third vertex of the triangle. So the triangle is determined by this information.

So if two angles and the side between them in one triangle are equal to two angles and the side between them in another triangle, then the triangles are congruent, as illustrated in Figure 38.

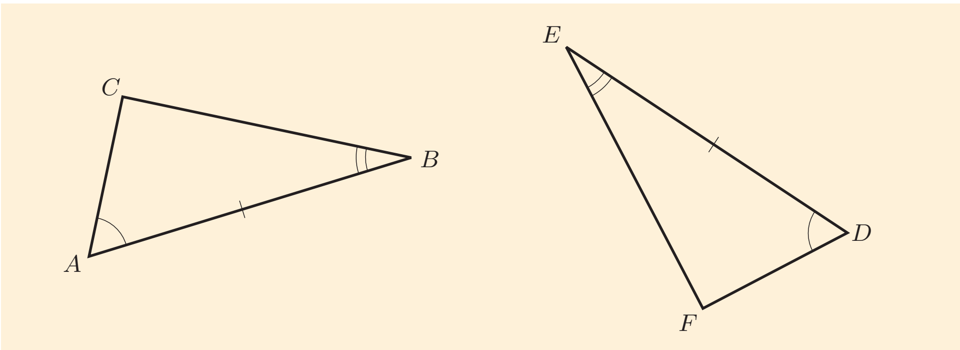


Figure 38 Two triangles that are congruent by ASA

The angle-angle-side case (AAS)

Consider the situation in which you know two angles of a triangle, as in the previous case, but instead of knowing the side *between* the angles, you know one of the other two sides. Since the third angle of the triangle can be calculated by using the fact that the sum of the angles is 180°, you still know two angles and the side between them, which is the previous case, and hence the triangle is determined.

So if you know that two angles and a side of a triangle in the order angle-angle-side are equal to two angles and a side of another triangle *in the same order*, then the triangles are congruent, as shown in Figure 39.

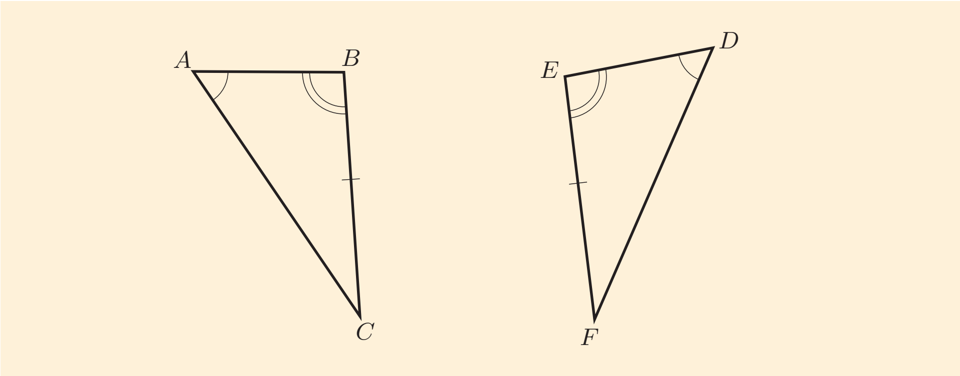


Figure 39 Two triangles that are congruent by AAS

You can check that the two triangles in Figure 39 have their equal angle-angle-side configurations in the same order by noticing that if you trace your finger around $\triangle ABC$ from the angle with one arc to the angle with two arcs, then the side with the stroke is next, and similarly if you trace your finger around $\triangle DEF$ from the angle with one arc to the angle with two arcs, then again the side with the stroke is next.

The angle-angle-angle case (AAA)

This is the situation where you know all three angles in a triangle. This is not enough to determine the triangle, because one triangle could be a scaled-up or scaled-down version of the other.

So if you know that the three angles in one triangle are the same as the three angles in another triangle, then you *cannot* conclude that the triangles are congruent.

The angle-side-side case (ASS)

If you know two sides and a non-included angle of a triangle, then there can be two possibilities for the triangle. This is illustrated in Figure 40, which shows that if you know the angle θ and the lengths of the sides a and b, then the third vertex F could be in either of the two positions shown.

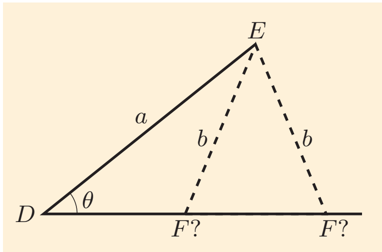


Figure 40 Two possible triangles if angle θ and lengths a and b are known

So if you know that two sides and a non-included angle of one triangle are equal to two sides and a non-included angle of another triangle, then you *cannot* conclude that the triangles are congruent.

Conclusion

All six possible cases of three pieces of information have now been covered. Four of them (SSS, SAS, ASA and AAS) show that two triangles are congruent, while the other two (AAA and ASS) do not guarantee congruency. These findings are summarised overleaf.

Strategy *To check whether two triangles are congruent*

Two triangles are congruent if one of the following situations occurs.

- The three sides of one triangle are equal to the three sides of the other triangle (SSS).
- Two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle (SAS).
- Two angles and the included side of one triangle are equal to two angles and the included side of another triangle (ASA).
- Two angles and a side of one triangle in the order angle-angle-side are equal to two angles and a side of the other triangle *in the same order* (AAS).

In fact, the ASA and AAS cases can be summarised as one criterion by using the idea of *corresponding sides*. If two triangles have the same three angles (as they do in the ASA and AAS cases), then a side in one triangle is said to **correspond** to a side in the other triangle if they are opposite equal angles. For example, in Figure 39 on page 33 the sides BC and EF are corresponding sides since they are opposite angles with one arc. Similarly, the sides AC and DF are corresponding sides, and the sides AB and DE are corresponding sides. The ASA and AAS cases can be summarised by saying that two triangles are congruent if two angles and a side of one triangle are equal to two angles and the *corresponding* side of the other triangle. You may prefer to use these conditions in this form.

Using the conditions for congruent triangles

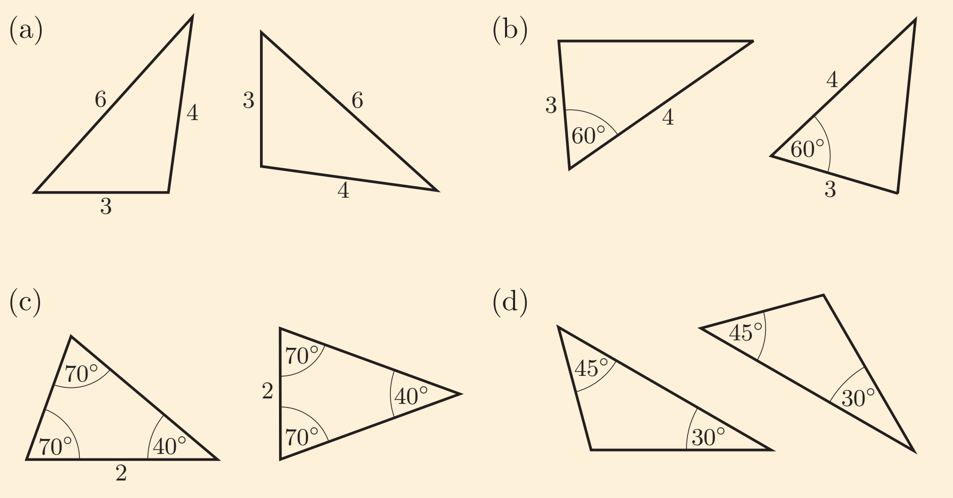
You can practise using the strategy above in the next activity.

Activity 21 *Checking whether triangles are congruent*

For each of the following pairs of triangles, determine (if possible) whether the two triangles are congruent, explaining your answers.

In parts (a)–(c) of this activity no particular units are specified for the lengths of the sides of the triangles. This is often done in geometry, just as when we use coordinate axes we often don't specify particular units for the numbers on the axis scales.

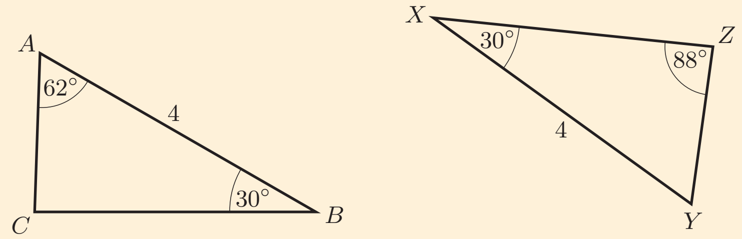
Also, the diagrams are not drawn to scale, since this activity requires you to use the conditions for congruency rather than measurement.



The next example illustrates that sometimes you need to find more angles before using one of the conditions to show that two triangles are congruent.

Example 7 *Showing that triangles are congruent*

Show that the triangles below are congruent.



Solution

Work out the unknown angle in $\triangle YXZ$, so that the angles of the two triangles can be compared.

The interior angles of a triangle add up to 180°. So

$$\angle ZYX = 180° - 88° - 30° = 62°.$$

Show that the triangles are congruent by using one of the conditions.

In $\triangle ABC$ and $\triangle YXZ$:

- $\angle CAB = \angle ZYX$ (both angles are 62°)
- $AB = YX$ (both sides have length 4)
- $\angle ABC = \angle YXZ$ (both angles are 30°).

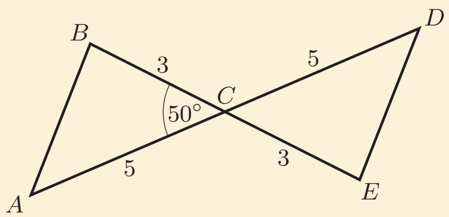
So $\triangle ABC \cong \triangle YXZ$ (by ASA).

The condition AAS could have been used here as an alternative to ASA.

Here is a similar activity for you to try.

Activity 22 *Showing that triangles are congruent*

Show that the two triangles in the diagram below are congruent.



If you can prove that two triangles are congruent, then you can deduce that corresponding angles and sides are equal. This can be a powerful method for establishing properties of shapes. For example, a kite can be defined as a quadrilateral with two pairs of adjacent equal sides. In Table 3, it was stated that there is also a pair of opposite equal angles, but how does that follow from the definition?

In the next example you will see how this can be deduced using congruent triangles. The example involves two triangles that share a side. A side like this is called a **common side** of the triangles.

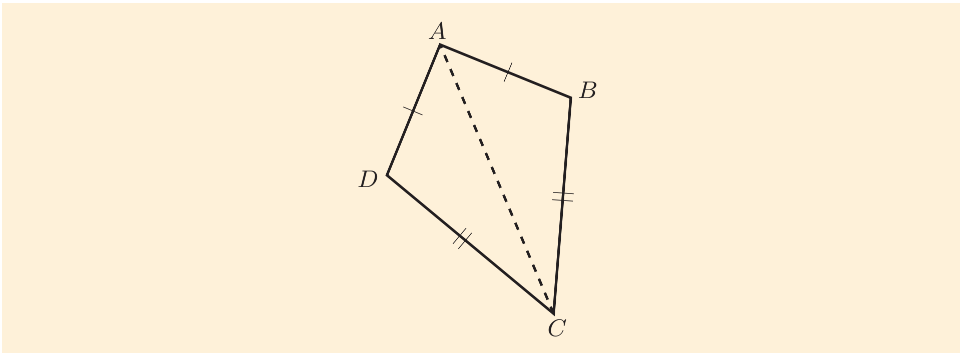
Example 8 *Proving that a kite has two opposite equal angles*

Use the fact that a kite is a quadrilateral with two pairs of adjacent equal sides to prove that it also has a pair of opposite equal angles.

Solution

Draw a kite with vertices labelled A, B, C, D. Mark the equal sides: $AB = AD$, $BC = DC$. Then draw the diagonal AC as a helpful construction. Show that the two resulting triangles are congruent.

The diagram below shows a kite with its two pairs of equal sides marked.



In $\triangle ABC$ and $\triangle ADC$:

- $AB = AD$ (given)
- $BC = DC$ (given)
- the side AC is common to both triangles.

So $\triangle ABC \cong \triangle ADC$ (by SSS).

Now use facts about congruent triangles.

Hence the corresponding angles of $\triangle ABC$ and $\triangle ADC$ are equal. The angles opposite AC are $\angle ABC$ in $\triangle ABC$ and $\angle ADC$ in $\triangle ADC$. So $\angle ABC = \angle ADC$.

That is, the kite has two opposite equal angles.

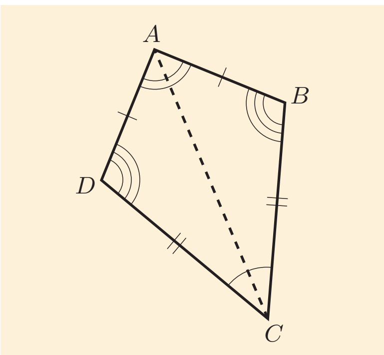


Figure 41 The kite in Example 7 with corresponding angles of congruent triangles marked

Congruent triangles can be used to prove even more properties of kites. For example, in Figure 41 the corresponding angles of the congruent triangles $\triangle ABC$ and $\triangle ADC$ are marked as equal. From this you can see another property: the diagonal AC **bisects** each of $\angle DAB$ and $\angle BCD$, that is, it cuts each of these angles into two equal parts. And this result leads on to further properties. If you draw the diagonal BD, congruent triangles can be used again to show that the diagonals cross each other at right angles. You might like to try this.

Congruent triangles can even help with a game of snooker! Figure 42 shows three balls on a snooker table. The player has to hit the white ball with the cue so that it strikes the red ball. The black ball is in the way, so this has to be achieved by bouncing the white ball off the cushion, as shown. The ball bounces off the cushion at the same angle as it strikes the cushion. These angles are both marked θ in Figure 42.

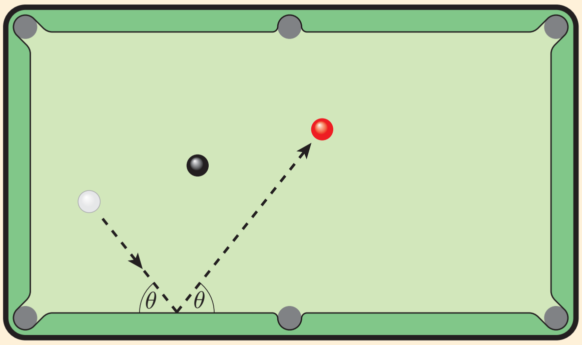
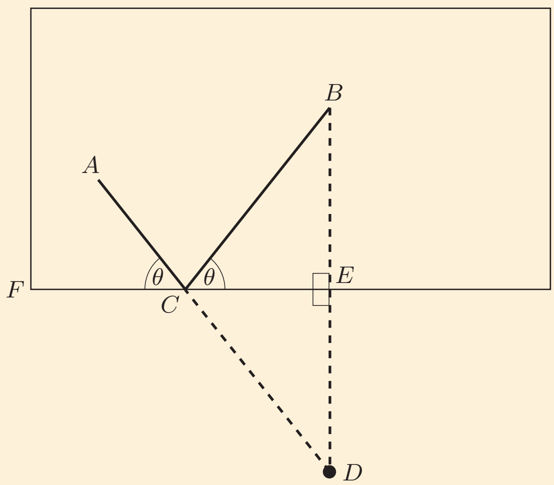


Figure 42 Balls on a snooker table

The player's problem is to decide the point where the white ball must strike the cushion if it is to hit the red ball. You are asked to look at the geometry of this problem in the next activity, and after the activity you will see that this gives the player a strategy for hitting the red ball.

Activity 23 *Investigating angles on a snooker table*

In the diagram below, the line indicating the initial path of the white ball has been extended, and a line has been drawn down from the position of the red ball, perpendicular to the cushion. The point D is the point where these two lines meet. As in Figure 42, the two equal angles between the cushion and the path of the white ball are both labelled as θ.



(a) Show that $\triangle BCE$ and $\triangle DCE$ are congruent.

(b) Deduce that the line segments BE and DE are equal.

So, from the solution to Activity 23, if the snooker player imagines the point that is the 'reflection' of the red ball in the cushion and hits the white ball in that direction, then it will bounce off the cushion and hit the red ball.

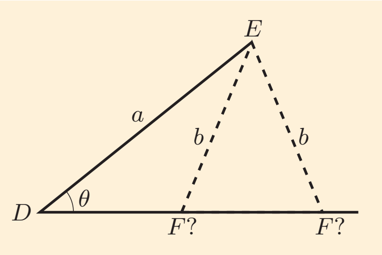


Figure 43 Two possible triangles if angle θ and lengths a and b are known

Finally in this subsection, let's briefly look at the ASS case again, which was one of the two cases that does not guarantee congruency. The diagram that illustrated this case is repeated in Figure 43. Remember that this diagram shows that if you know angle θ and the lengths of the sides a and b, then there could be two possibilities for the triangle, since the third vertex F could be in either of the two positions shown.

However if you also know that angle θ is a *right angle* or an *obtuse angle*, then there is only one possibility for the triangle. So if you know that two sides and a non-included angle of one triangle are equal to two sides and a non-included angle of another triangle, in the same order, and you also know that this angle is $90°$ or greater, then you *can* say that the triangles are congruent.

In this subsection you have used different conditions to show that two triangles are congruent, and you have used the congruence of triangles to deduce further results. In the next subsection, the same sort of approach is used to determine when two triangles are similar and how this property can be used in practice.

3.2 Similar triangles

Similar triangles are the same shape but not necessarily the same size. As with congruence, the triangles may be rotated or flipped over. For example, the two triangles in Figure 44 are similar to each other.

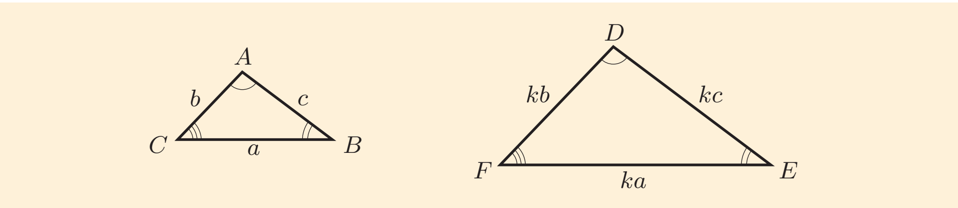


Figure 44 Two similar triangles

Any two similar triangles have the following properties:

- the three angles of one triangle are equal to the three angles of the other triangle;
- the three sides of one triangle are *in proportion* to the three sides of the other triangle.

The second property means that the three side lengths of one triangle are obtained from the three side lengths of the other triangle by multiplying by the same scale factor. In Figure 44 the scale factor from the first triangle to the second triangle is k.

Another way to think about the second property is that it means that the ratios of the sides of the two triangles are equal. For the triangles in Figure 44,

$$\frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA}, \qquad (2)$$

since all these ratios are equal to k.

Alternatively, you can write
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD},$$
since all these ratios are equal to $1/k$.

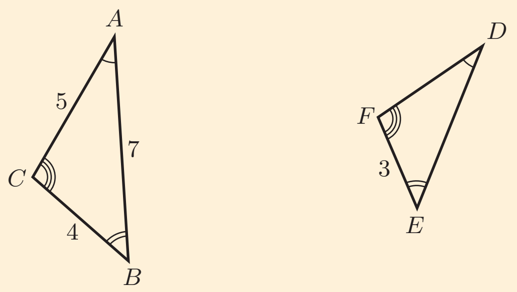
To check whether two triangles are similar, you can check either of the two properties immediately below Figure 44 – the other property then holds automatically.

So if you know that two triangles have the same three angles, then they are similar and so they also have their sides in proportion. (This also holds if two angles in one triangle are equal to two angles in the other triangle, since then the third angles must also be equal.) This means that you can immediately write down equations like equations (2) for the triangles. The numerator and denominator of each equal ratio are sides that are opposite equal angles; that is, they are *corresponding sides*. Writing down these ratios is a useful way to find the lengths of unknown sides in similar triangles, as illustrated in the example below.

Similarly, if you know that two triangles have their sides in proportion, then they are similar and so they also have three equal angles. You can tell which angles are equal by using the fact that the sides on the numerator and denominator of each equal ratio are corresponding sides and therefore opposite equal angles.

Example 9 *Finding unknown side lengths in similar triangles*

Find the length of the side DE in the diagram below.



Solution

The diagram indicates that the two triangles have the same three angles, so they are similar.

Also, the sides BC and EF correspond, because they are opposite the angles marked with one arc, and similarly AC and DF correspond, and AB and DE correspond. Hence

$$\frac{EF}{BC} = \frac{DF}{AC} = \frac{DE}{AB}.$$

In particular,

$$\frac{EF}{BC} = \frac{DE}{AB}.$$

Substituting in the known lengths gives

$$\frac{3}{4} = \frac{DE}{7}.$$

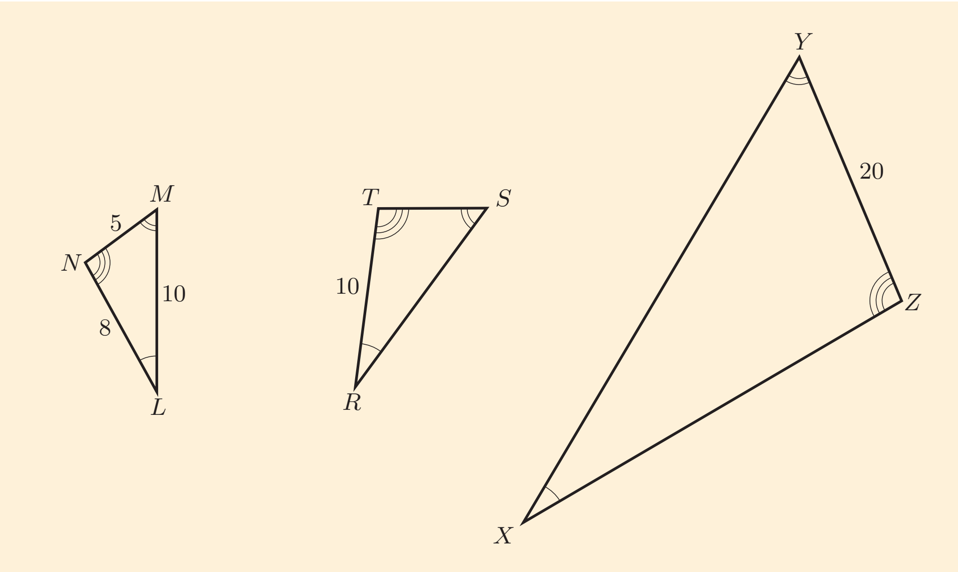
Hence

$$DE = \frac{3 \times 7}{4} = \frac{21}{4} = 5\tfrac{1}{4}.$$

Here is a similar activity for you to try.

Activity 24 *Finding unknown side lengths in similar triangles*

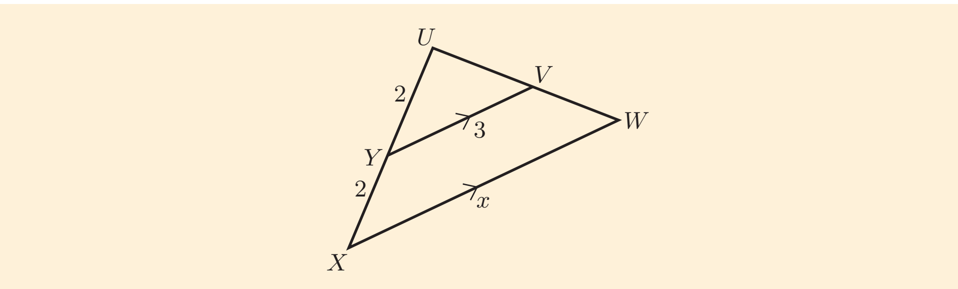
The triangles below are similar. Calculate the lengths TS, RS, XZ and YX.



Angles or lengths in geometric diagrams can often be deduced by finding and using similar triangles. This is illustrated in the next example.

Example 10 *Finding an unknown side in similar triangles*

Find the length x in the diagram below.



Solution

Show that $\triangle UYV$ and $\triangle UXW$ are similar.

The line segments YV and XW are parallel. So in $\triangle UYV$ and $\triangle UXW$:

- $\angle UYV = \angle UXW$ (these are corresponding (F) angles)
- $\angle UVY = \angle UWX$ (these are corresponding (F) angles).

Since two angles in one triangle are equal to two angles in the other, $\triangle UYV$ is similar to $\triangle UXW$.

Note that $\angle YUV$ is common to both triangles, and this fact could have been used instead of one of the facts here.

Write down the ratio of corresponding sides, preferably with the unknown side in the numerator.

The sides YV and XW are corresponding, since they are both opposite the common angle, $\angle YUV$. Similarly, UY and UX are corresponding, as are UV and UW. Hence

$$\frac{XW}{YV} = \frac{UX}{UY} = \frac{UW}{UV}.$$

Now, $UY = 2$ and $YX = 2$, so $UX = 2 + 2 = 4$. So

$$\frac{4}{2} = \frac{x}{3},$$

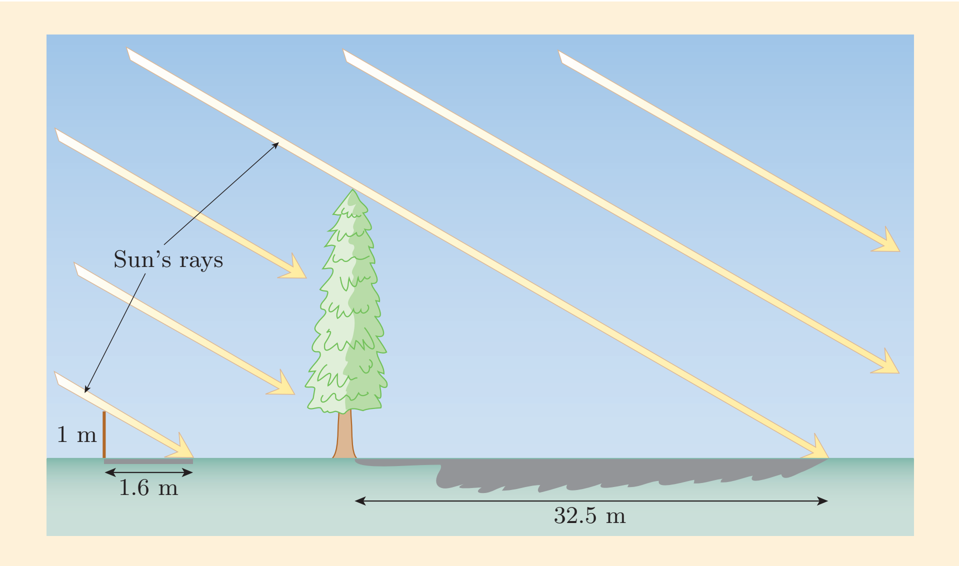
and hence

$$x = \frac{4 \times 3}{2} = 6.$$

Similar triangles can be used to determine lengths that cannot be easily measured, as the next activity illustrates. The activity shows you how to measure the height of a tree without having to climb a ladder with a tape measure. This method of measuring the heights of tall objects was used extensively in the sixteenth and seventeenth centuries!

Activity 25 *Finding the height of a tree*

Suppose that on a sunny day you place a stick in the ground near a tree, so that the length of the exposed part of the stick is 1 m. You then measure the lengths of the shadows cast by the stick and the tree, as shown below. The stick is placed at the same angle as the tree (in the diagram they are both vertical), and the ground is level.



(a) Assuming that the rays of the Sun are parallel, show that the two triangles in the diagram are similar. (One triangle is formed by the stick, its shadow on the ground and a ray of the Sun; the other is formed by the tree, its shadow on the ground and a ray of the Sun.)

(b) Suppose that the length of the stick's shadow is 1.6 m and the length of the tree's shadow is 32.5 m. Calculate the height of the tree.

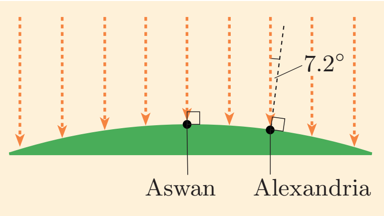


Figure 45 The Sun's rays at the same time at Aswan and Alexandria

The fact that the Sun's rays are parallel was used by Eratosthenes (who was mentioned in Unit 3) to estimate the size of the Earth. He knew (by observing shadows) that at noon at midsummer, the Sun's rays were vertical at Aswan in Egypt and at the same time were at an angle of 7.2° to the vertical at Alexandria, also in Egypt (Figure 45). The angle 7.2° is $\frac{1}{50}$ of a full turn, so he concluded that the distance from Alexandria to Aswan was $\frac{1}{50}$ of the circumference of the Earth. His estimate was remarkably accurate, but historians argue about how accurate, because no one is sure about the size of Eratosthenes' unit of length, the *stadion*.

Checking for similarity

There is a third useful way of showing that two triangles are similar. Suppose that two triangles have the property shown in Figure 46. They have one equal angle and the sides containing this angle are in proportion.

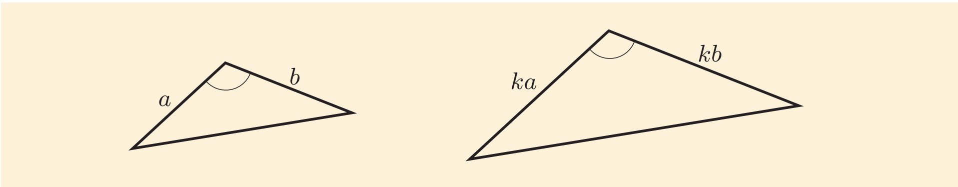


Figure 46 Two triangles with one equal angle and the sides containing this angle in proportion

This property guarantees that the triangles are similar. To see this, imagine scaling the first triangle in Figure 46 by the scale factor k. The scaled triangle will be similar to the first triangle, but it will also be congruent to the second triangle, by SAS. So the two triangles in Figure 46 are similar.

So you now have three ways to check whether two triangles are similar.

Strategy *To check whether two triangles are similar*

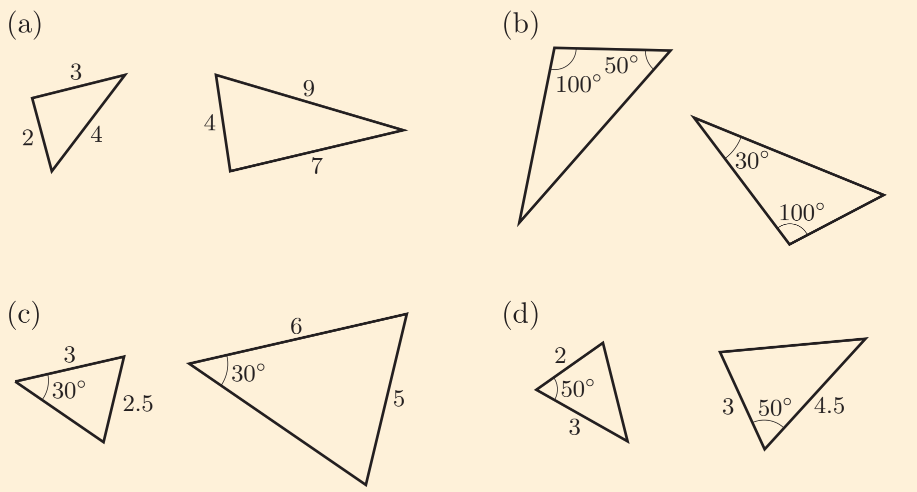
Two triangles are similar if any one of the following three conditions holds. Then the other two conditions also hold.

- The three angles of one triangle are equal to the three angles of the other triangle.
- The three sides of one triangle are in proportion to the three sides of the other triangle (their ratios are equal).
- An angle of one triangle is equal to an angle of the other triangle, and the sides containing these angles are in proportion (their ratios are equal).

Try using these conditions to see if you can spot which triangles are similar in the activity on the next page.

Activity 26 *Checking for similarity*

Which of the following pairs of triangles are pairs of similar triangles? Assume only that they have the properties marked.



In the next subsection you will see how similar triangles can be used to prove one of the most famous theorems in mathematics.

3.3 Pythagoras' Theorem

Pythagoras' Theorem is one of the oldest mathematical results known; it involves the sides of a right-angled triangle. The longest side, which is always the side opposite the right angle, is called the **hypotenuse**. This is the side AB in Figure 47.

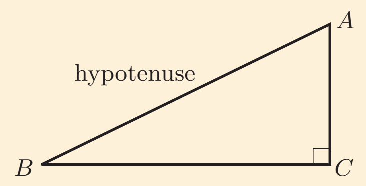


Figure 47 A right-angled triangle

Pythagoras' Theorem

For a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

For example, for the right-angled triangle in Figure 47,

$$AB^2 = AC^2 + BC^2.$$

AB^2 is the notation used for the square of the length AB.

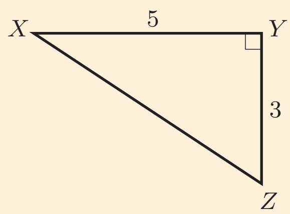
Pythagoras' Theorem has long been attributed to Pythagoras, a Greek of the sixth century BC, who gave his name to a sect called the Pythagoreans. (Pythagoras' existence is disputed by many historians, however.) The Pythagoreans believed that numbers and number patterns were the key to understanding the world. However, it is clear from clay tablets dating from about 2000 BC that early Babylonian scribes knew about the theorem, and the result is also found in ancient Chinese manuscripts. Pythagoras' Theorem is Proposition 47 of Book 1 of Euclid's *Elements*, and this is where the first rigorous proof appears.

Pythagoras' Theorem is usually read as *Pythagoras's Theorem*.

Later in this subsection you will see why Pythagoras' Theorem is true. First we look at how it can be used to calculate the third side of a right-angled triangle when two sides are already known.

Example 11 *Using Pythagoras' Theorem to find the hypotenuse*

Calculate the length of the hypotenuse of the triangle below.



Solution

Relate the diagram to the statement of Pythagoras' Theorem.

XZ is the hypotenuse and XY and YZ are the shorter sides.

Now use the theorem.

By Pythagoras' Theorem,

$$XZ^2 = XY^2 + YZ^2.$$

Substituting in the lengths of the known sides gives

$$XZ^2 = 5^2 + 3^2 = 25 + 9 = 34.$$

This gives the length of the hypotenuse XZ as $\sqrt{34}$. (The alternative solution $-\sqrt{34}$ is rejected because lengths must be positive.)

In Example 11 the answer was given as a surd, $\sqrt{34}$. This is acceptable because the question is an abstract geometric problem. However, if the question were 'How far from your starting point would you be if you walked 3 metres north followed by 5 metres west?', then a decimal answer would be more appropriate. In this case the answer 'about 5.8 metres' would be sensible.

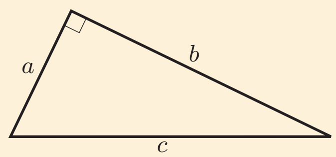


Figure 48 A right-angled triangle with shorter sides a and b, and hypotenuse c

Sometimes it is convenient to use Pythagoras' Theorem for a right-angled triangle with labelled sides rather than labelled vertices. For example, Pythagoras' Theorem applied to the right-angled triangle in Figure 48 gives

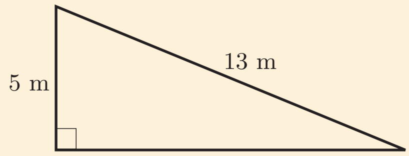
$$c^2 = a^2 + b^2.$$

When you know two side lengths of a right-angled triangle, they are not always the two sides adjacent to the right angle. The next example shows how to proceed in a case like this.

Example 12 *Using Pythagoras' Theorem to find a shorter side*

Tutorial clip

Calculate the length of the third side of the triangle below.



Solution

Let the length of the unknown side be b m. By Pythagoras' Theorem,

$$13^2 = 5^2 + b^2.$$

So

$$b^2 = 13^2 - 5^2 = 169 - 25 = 144.$$

Since b represents a length, we take the positive square root, which gives

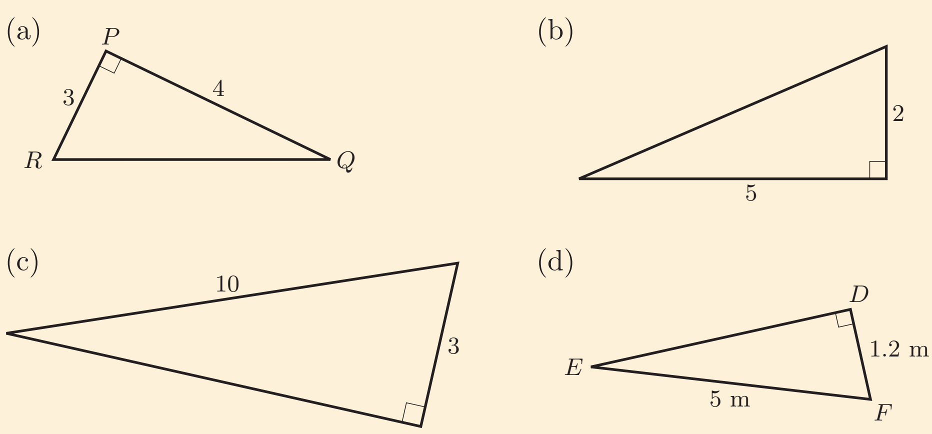
$$b = \sqrt{144} = 12.$$

So the length of the third side is 12 m.

You can practise using Pythagoras' Theorem in the next activity.

Activity 27 *Using Pythagoras' Theorem*

Calculate the length of the third side of each of the following right-angled triangles. In part (d), give your answer to two decimal places.



These diagrams are purposely not drawn to scale: this activity requires calculation rather than measurement.

In Activity 27(a) the right-angled triangle has whole-number lengths for the three sides, namely 3, 4 and 5. So

$$3^2 + 4^2 = 5^2.$$

Three whole numbers (like 3, 4 and 5) such that the square of one of them is equal to the sum of the squares of the other two are said to form a **Pythagorean triple**.

The Italian mathematician Leonardo Fibonacci (1170–1250) gave the following method for finding Pythagorean triples in one of his books.

Take any odd square number. Add up all the odd numbers that are smaller than this number; this will give another square number (you saw this fact in Unit 1). When you add the two square numbers together, you always get a third square number (this follows from the same fact in Unit 1). That is, you've found a Pythagorean triple.

There are other Pythagorean triples besides $3, 4, 5$. For example, Example 12 shows that $5, 12, 13$ is a Pythagorean triple. In fact, there are infinitely many Pythagorean triples – a method for calculating them is described in Euclid's *Elements*.

Pythagorean triples can be used to construct right angles, because the converse of Pythagoras' Theorem is true. It can be stated as follows.

If a triangle has sides of lengths a, b and c with $a^2 + b^2 = c^2$, then the angle opposite the side of length c is a right angle.

There is strong evidence that the $3, 4, 5$ Pythagorean triple was known to the ancient Egyptians and Babylonians long before Pythagoras. There is also evidence that the ancient Egyptians knew that the converse of Pythagoras' Theorem is true and used it in practical situations to construct right angles; it is thought that they constructed $3, 4, 5$ triangles using knots on a string in order to obtain accurate right angles.

There are many ways of proving Pythagoras' Theorem – the book *The Pythagorean Proposition* by E.S. Loomis (published in 1968) collects and classifies 370 proofs.

Now we move on from using Pythagoras' Theorem to the problem of showing why it is true. The video in the following activity demonstrates a visual proof of Pythagoras' Theorem.

Video

Activity 28 *Proof of Pythagoras' Theorem*

Watch the video *Proof of Pythagoras' Theorem.*

In addition to the visual proof shown in the video, there are more formal algebraic proofs. To end this section there follows an algebraic proof using the idea of similar triangles.

Proof of Pythagoras' Theorem

Consider the right-angled triangle shown in Figure 49.

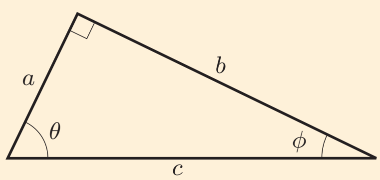


Figure 49 A right-angled triangle

Split the triangle by drawing a perpendicular from the vertex at the right angle. This gives the two triangles shown in Figure 50. The angles θ and ϕ are unchanged from Figure 49.

A **perpendicular** is a line at right angles to a given line.

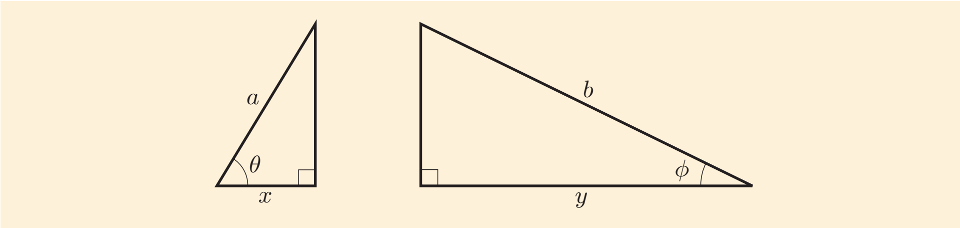


Figure 50 The triangle in Figure 49 split by the perpendicular from the right angle

The hypotenuse c has been split into two lengths x and y, so

$$c = x + y. \tag{3}$$

Now we express x and y in terms of the original lengths a, b and c.

First consider the left-hand triangle in Figure 50. Figure 51 shows this triangle flipped and rotated to be in the same orientation as the original triangle in Figure 49. It is similar to the original triangle (because they both have angles θ and 90°). So the ratios of corresponding sides are equal. Hence

$$\frac{x}{a} = \frac{a}{c}.$$

Multiplying both sides by a gives

$$x = \frac{a^2}{c}. \tag{4}$$

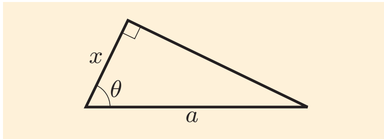


Figure 51 The left-hand triangle in Figure 50 flipped and rotated to be in the same orientation as the original triangle in Figure 49

Now consider the right-hand triangle in Figure 50. Figure 52 shows this triangle flipped and rotated to be in the same orientation as the original triangle in Figure 49. This new triangle is also similar to the original triangle (because they both have angles ϕ and 90°). Again the ratios of corresponding sides are equal, so

$$\frac{y}{b} = \frac{b}{c}.$$

Multiplying both sides by b gives

$$y = \frac{b^2}{c}. \tag{5}$$

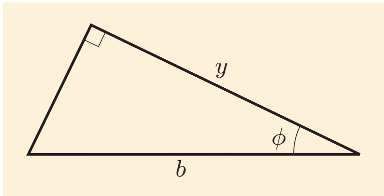


Figure 52 The right-hand triangle in Figure 50 flipped and rotated to be in the same orientation as the original triangle in Figure 49

Using equations (4) and (5) to substitute into equation (3) gives

$$c = \frac{a^2}{c} + \frac{b^2}{c}.$$

Multiplying through by c gives

$$c^2 = a^2 + b^2.$$

This is Pythagoras' Theorem.

4 Perimeters and areas

A shape that can be drawn in a plane, such as the triangles and other polygons introduced in Section 2, is called a **plane shape**. This section considers two questions about plane shapes:

- What is the distance around the boundary of a shape – that is, what is its *perimeter*?
- How much surface does a shape occupy – that is, what is its *area*?

For help with areas and perimeters, see Maths Help Module 7, Section 3.

The section revises these properties for some basic shapes. It also discusses an application of these ideas: you will see how they can sometimes be used to determine whether cells in tissue samples are abnormal.

4.1 Finding perimeters and areas

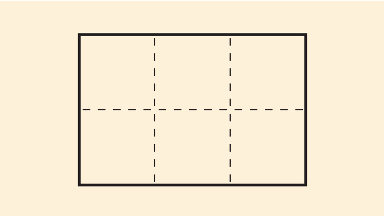


Figure 53 A rectangle cut into squares

The distance around the boundary of a shape is known as its **perimeter**. For example, Figure 53 shows a rectangle that is 2 cm by 3 cm. Its perimeter is

$$(3 + 2 + 3 + 2)\,\text{cm} = 10\,\text{cm}.$$

The notion of **area** derives from the idea of counting how many squares of a standard size are needed to 'make' a shape. For example, the rectangle in Figure 53 can be cut into six squares, each with sides of length 1 cm, as shown. The area of each square is 1 *square* centimetre (written $1\,\text{cm}^2$). So the figure shows that the area of the 2 cm by 3 cm rectangle is

$$(2 \times 3)\,\text{cm}^2 = 6\,\text{cm}^2.$$

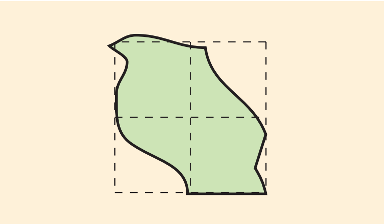


Figure 54 An irregular shape covering about two whole squares and two half-squares

Similarly, Figure 54 shows an irregular shape whose area is approximately

$$\left(1 + 1 + \tfrac{1}{2} + \tfrac{1}{2}\right)\,\text{cm}^2 = 3\,\text{cm}^2.$$

If you wish to measure large areas, then square metres (m^2) or square kilometres (km^2) may be more appropriate units than square centimetres. For example, one square metre is the area of a square that is one metre by one metre.

You can convert between units of area by considering how many of the smaller units fit into one of the larger units. For example, a square of side 1 m contains $100 \times 100 = 10\,000$ squares of side 1 cm:

$$\begin{aligned} 1\,\text{m}^2 &= (1\,\text{m}) \times (1\,\text{m}) \\ &= (100\,\text{cm}) \times (100\,\text{cm}) \\ &= 10\,000\,\text{cm}^2. \end{aligned}$$

So to convert a measurement in m^2 to cm^2, you multiply by 10 000. For example, an area of $0.5\,\text{m}^2$ is the same as

$$(0.5 \times 10\,000)\,\text{cm}^2 = 5000\,\text{cm}^2.$$

There are simple formulas for the areas of many standard shapes. The rectangle in Figure 53 illustrates the following formula.

> A rectangle with sides a and b has area ab.

The rectangle in Figure 53 has short sides of length 2 cm and long sides of length 3 cm, and its area is $(2 \times 3)\,\text{cm}^2 = 6\,\text{cm}^2$.

When you use this formula, or any of the other formulas for area that you will meet in this subsection, you must be careful to use consistent units. For example, if a is in cm, then b must also be in cm, and the answer for the area will then be in cm^2.

The formula for the area of a rectangle can be used to find a formula for the area of a parallelogram. The **base** of a parallelogram can be taken to be any of its sides, and its **perpendicular height** is then its height measured at right angles to the base. The usual convention is to choose the 'bottom' side as the base. For example, the parallelogram in Figure 55 has base b and perpendicular height h.

In general, two lines are perpendicular if they meet or cross at right angles.

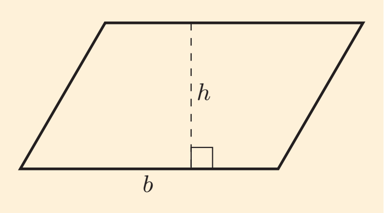


Figure 55 The base and perpendicular height of a parallelogram

Now consider splitting a parallelogram and reassembling it as shown in Figure 56. A right-angled triangle is cut from the right of the shape and attached to the left.

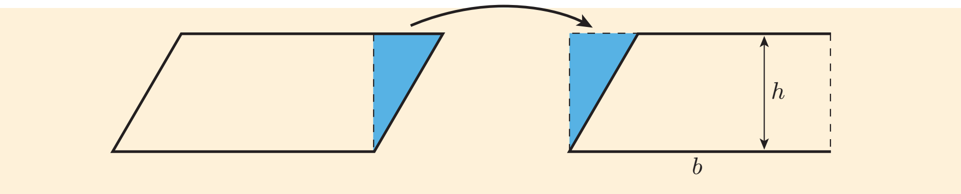


Figure 56 Splitting and reassembling a parallelogram

The triangle cut from the right of the parallelogram fits exactly on the left to form a rectangle, which gives the following formula.

A parallelogram with base b and perpendicular height h has area bh.

This result can in turn be used to derive a formula for the area of a triangle, in terms of its base and perpendicular height. As for a parallelogram, the base of a triangle can be taken to be any of its sides, and its perpendicular height is then its height measured at right angles to the base, as shown in Figure 57.

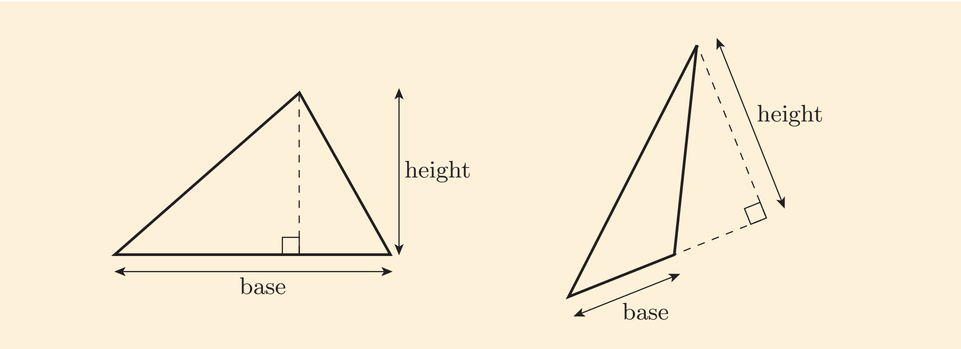


Figure 57 Base and height measurements of two triangles

Now consider making a copy of a triangle and rotating it through 180°, as shown in Figure 58(a).

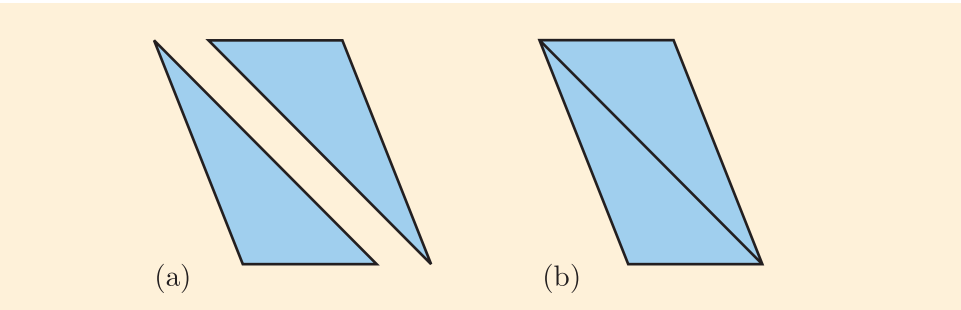


Figure 58 Putting together two copies of a triangle to form a parallelogram

Rotating a line through 180° leaves its direction unchanged, so each side of the original triangle is parallel to the corresponding side in the copy. So the triangle and the copy can be put together to form a parallelogram, as shown in Figure 58(b).

The area of the triangle is half of the area of the parallelogram, which gives the following formula.

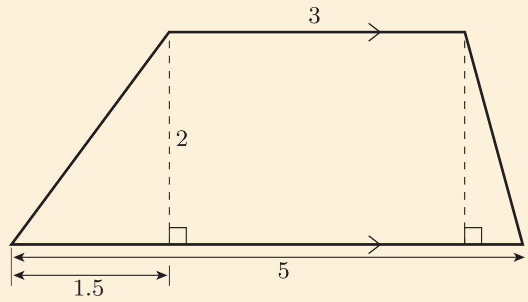
The area of a triangle with base b and perpendicular height h is $\frac{1}{2}bh$.

The formulas in this subsection can be used to find not only the areas of the given shapes, but also the areas of shapes that are combinations of them, as the next example illustrates.

In this example no particular units are specified for the lengths of the sides of the shape.

Example 13 *Finding the area of an irregular shape*

Find the area of the trapezium shown below.



Solution

Split the shape into simpler shapes.

This shape is split by the dashed lines into a rectangle and two triangles.

Find the areas of the simpler shapes.

On the left is a triangle with base 1.5 and perpendicular height 2, which has area $\frac{1}{2} \times 1.5 \times 2 = 1.5$.

In the middle is a 3 by 2 rectangle, which has area $3 \times 2 = 6$.

On the right is a triangle with base $5 - 1.5 - 3 = 0.5$ and perpendicular height 2, which has area $\frac{1}{2} \times 0.5 \times 2 = 0.5$.

So the total area of the shape is $1.5 + 6 + 0.5 = 8$.

An alternative way to find the area of the trapezium in Example 13 is to split it into two triangles, as shown in Figure 59. Each triangle has height 2 units, and the bases are 5 units and 3 units. So the area of the trapezium is

$$\frac{1}{2} \times 5 \times 2 + \frac{1}{2} \times 3 \times 2 = 5 + 3 = 8.$$

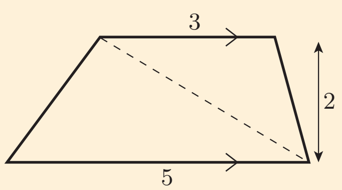


Figure 59 The trapezium in Example 13 split into two triangles

This method can be used for any trapezium, as illustrated in Figure 60.

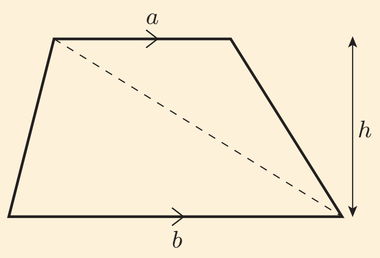


Figure 60 A trapezium split into two triangles

So the area of a trapezium with parallel sides a and b and perpendicular height h is given by the formula

$$\tfrac{1}{2}ah + \tfrac{1}{2}bh.$$

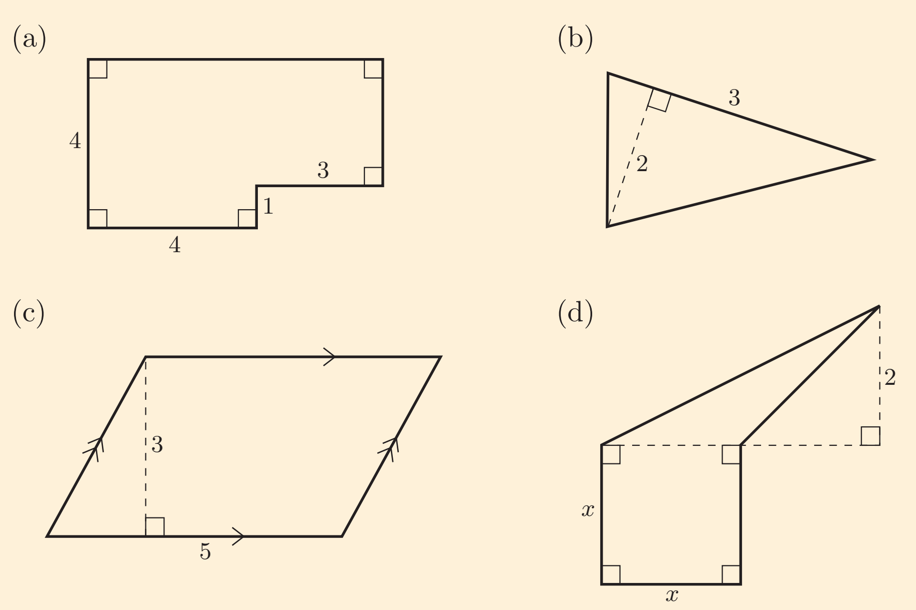
This expression can be simplified by taking out the common factor $\frac{1}{2}h$, which gives the formula below.

The area of a trapezium with parallel sides a and b and perpendicular height h is $\frac{1}{2}(a + b)h$.

You can practise finding the areas of shapes in the following activity.

Activity 29 *Finding areas*

Find the areas of the following shapes. In part (d), give an answer in terms of x.



4.2 Using perimeters and areas

An interesting medical application of geometry is the detection of abnormal cells in the body. Traditional methods for detecting abnormality involve a trained technician using a microscope to identify and count abnormal cells in a tissue sample, which is a tiring task. Counting is an ideal task for a computer, but how should the computer decide which cells are abnormal?

One general feature of the type of abnormal cells shown in Figure 61 is that they have a more 'spiky' boundary. But how can a computer be programmed to recognise this property?

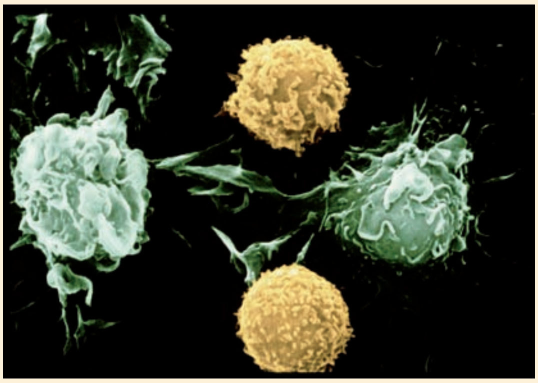


Figure 61 A photograph showing two normal cells (top and bottom) and two abnormal cells (left and right). The normal cells have much smoother surfaces.

The word 'pixel' comes from amalgamating the words *picture* and *element*.

Let's look at how the cells are presented to a computer. A microscope slide of cells is photographed to turn it into a computer image. A computer image is made up of *pixels*, which are small square regions of colour. Figure 62 illustrates this – it shows an image of two cells, a normal one on the left and an abnormal one on the right, with the size of the pixels exaggerated to make their square nature apparent.

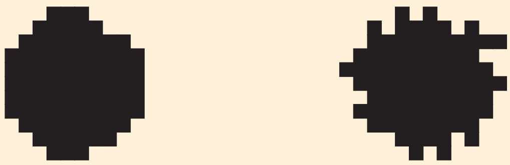


Figure 62 Two cells seen as collections of pixels

In practice this is the most difficult step of the process: successfully differentiating cells from the surrounding fluid.

The image in Figure 62 has also been 'thresholded'; that is, the interior of each cell has been coloured with a single colour, black, that is different from the colour of the surrounding area, which is beige. The question remains as to how a computer could classify the two cells in Figure 62 in terms of the spikiness of their boundaries.

It's easy for a computer to calculate the perimeters of the two cells, simply by counting the number of units around the boundary. The two cells in Figure 62 have perimeters of 42 units and 62 units, respectively. So the abnormal cell has a larger perimeter: this corresponds to the intuitive notion that the boundary of an abnormal cell is more spiky than that of a normal cell. However, a normal cell that is larger than the cells in

Figure 62 will also tend to have a larger perimeter. For example, the image in Figure 63 is that of a normal cell, but its perimeter is 62 units, the same as that of the abnormal cell in Figure 62. So measuring the perimeter alone cannot distinguish between normal and abnormal cells.

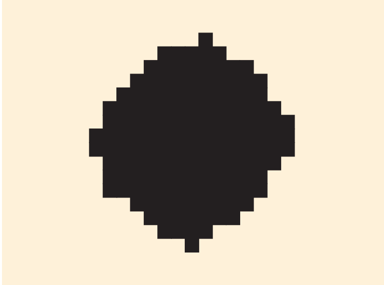


Figure 63 A larger normal cell

This leads to the idea of dividing the perimeter of a cell by its area to compensate for the size of the cell. A computer can calculate the area of a cell by counting the number of black squares – both cells shown in Figure 62 have area 83 square units. Compensating for size in this way leads to the measure that this module will call **wiggliness**, which is defined as follows.

$$\text{wiggliness} = \frac{\text{perimeter}^2}{\text{area}}$$

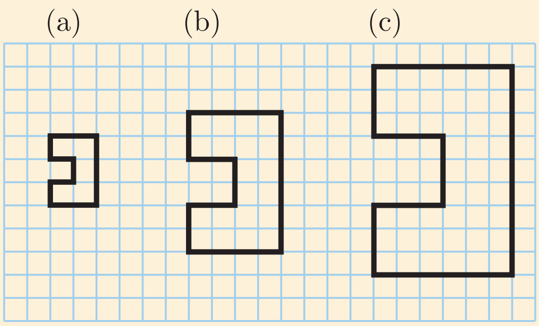
You may wonder why the simpler formula

$$\text{wiggliness} = \frac{\text{perimeter}}{\text{area}}$$

isn't used – why should the perimeter be *squared* in the formula? To see why, consider what happens when a shape is enlarged, as in the following activity.

Activity 30 *Calculating the wiggliness of enlarged shapes*

Calculate the area, perimeter and wiggliness of each of the three shapes below; give the wiggliness to one decimal place.



In this diagram each grid square has sides of length 1 unit.

The three shapes in Activity 30 are similar, with the second and third shapes enlarged compared to the first one. The scale factors of shapes (b) and (c) with respect to shape (a) are 2 and 3, respectively. When a shape is scaled, its perimeter, like all its lengths, is multiplied by the scale factor, so the perimeter of shape (b) is twice the perimeter of shape (a), and the perimeter of shape (c) is three times the perimeter of shape (a). However, the effect of scaling on its area is different. Figure 64 illustrates the effect of scaling on the area of a square whose sides are 1 unit long. If the length of the side is doubled, the area is four times as large, and if the side length is tripled, the area is nine times as large. In general, if the scale factor is k, then the area of the scaled square is k^2 times the area of the original square. Since the diagram in Activity 30 is made up of squares, the area of shape (b) is four times the area of shape (a), and the area of shape (c) is nine times the area of shape (a), as you can check from your solution.

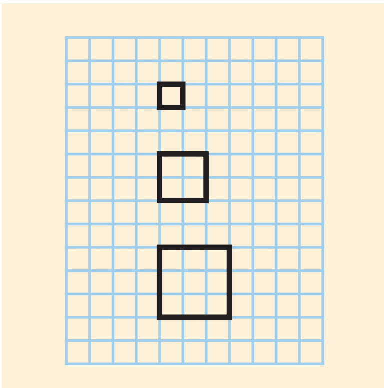


Figure 64 Area scale factors

These effects of enlargement on perimeter and area hold for any shape. If a shape is scaled by the scale factor k, then its perimeter is multiplied by k and its area is multiplied by k^2.

Because of this, the wiggliness of a shape, as defined in the formula in the pink box, is unchanged as the shape is enlarged. To see this, consider a shape with perimeter P and area A. It has wiggliness P^2/A. Any scaled version of the shape has perimeter kP and area k^2A for some constant k, so it has wiggliness

$$\frac{(kP)^2}{k^2A} = \frac{k^2P^2}{k^2A} = \frac{P^2}{A},$$

which is the same as for the original shape. This is precisely what we want: the wiggliness should be a property of the shape of an object that is independent of its size.

If the lengths are measured in metres, then the perimeter will also be measured in metres. So the perimeter squared will be measured in m^2, the same units as area. So the units will cancel, which means that 'wiggliness' is a number without units: its value is the same no matter what units are used to measure the lengths.

Wiggliness does not have any units associated with it: it is a 'pure number'. Such quantities are called **dimensionless quantities**, and they are often important in the investigation of real-world problems.

The next activity asks you to calculate the wiggliness of the three cell images in Figures 62 and 63.

Activity 31 *Calculating the wiggliness of cells*

(a) The large cell shown in Figure 63 has area 150 square units, so from the discussion above you now have the areas and perimeters of all three cells shown in Figures 62 and 63. Use this information to complete the following table; give the wiggliness to one decimal place.

Cell	Area	Perimeter	Wiggliness
Small normal			
Abnormal			
Large normal			

(b) Using the above table, suggest a criterion for determining whether a cell is abnormal.

4.3 Circles and π

Strictly, the circle is *just* the points on the boundary, and the 'filled-in' shape is called a *disc*. However, in this module we will use 'circle' to mean the whole shape.

The plural of radius is *radii*.

A **circle** is a shape whose boundary consists of all the points that are a fixed distance from a fixed point called the **centre** – to see that this is true, think about how you would draw a circle using a pair of compasses.

Special names are given to some parts of a circle, as illustrated in Figure 65. The boundary of a circle is called its **circumference**. A line segment from the centre to the circumference of a circle is called a **radius**. A line segment starting and ending on the circumference is called a **chord**. A chord that passes through the centre of the circle is called a **diameter**. Any unbroken section of the circumference is called an **arc**. The shape enclosed by an arc of a circle together with the two radii from the endpoints of the arc is called a **sector**. A **segment** is the shape enclosed by an arc and the chord joining the ends of the arc.

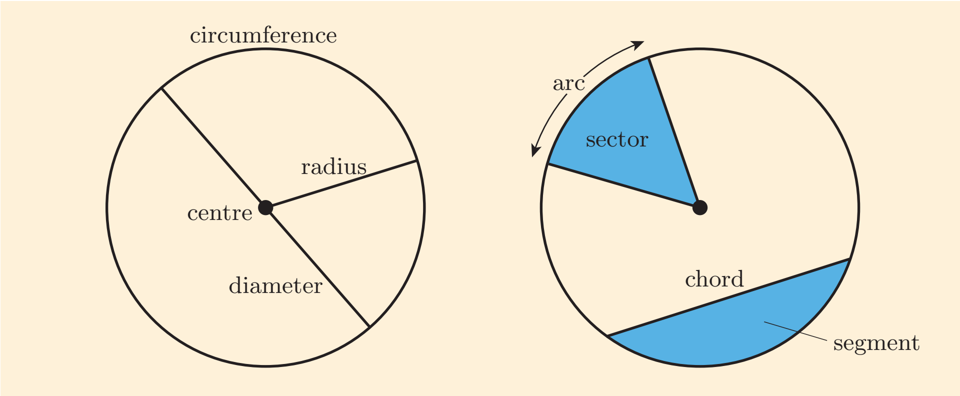


Figure 65 Parts of circles

The words circumference, radius and diameter are also used to refer to the *lengths* of these parts of a circle. For example, you can say that a circle has radius 2 cm.

The shape enclosed by a diameter of a circle, together with one of the two arcs from one end of the diameter to the other, is called a **semicircle** (Figure 66). Thus a semicircle is both a sector and a segment.

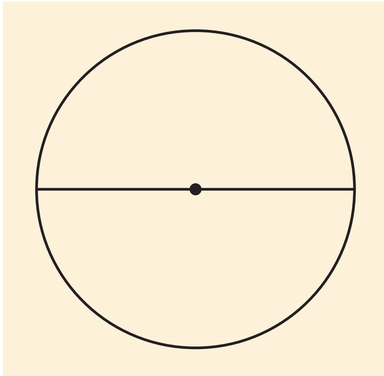


Figure 66 A diameter divides a circle into two semicircles

There is a well-known relationship between the circumference and radius of a circle:

> The circumference of a circle of radius r is $2\pi r$.

The number π is a constant that is one of the most remarkable numbers in mathematics, as it occurs frequently in many different contexts. It is a number that cannot be written down exactly as a terminating or recurring decimal – that is, it is irrational – so approximations to π are needed for practical applications. The value of π is 3.141 592 65 to eight decimal places.

Unit 3 discussed the fact that every rational number has an expression as a terminating or recurring decimal.

There has been a lot of effort expended in calculating approximations to π.

In ancient times π was frequently taken to be 3, such as in the following Bible extract that describes the building of part of the temple of Solomon.

> Then he made the Sea of cast bronze, ten cubits from one brim to the other; it was completely round. Its height was five cubits, and a line of thirty cubits measured its circumference.
>
> (*New King James Bible*, II Chronicles 4:2)

The cubit is one of the earliest recorded units of length. Its length varied between different cultures – in Egypt, the royal cubit was about 52 cm.

From this quotation, the diameter of the object (equal to twice the radius) is ten cubits and the circumference is thirty cubits. Dividing the circumference by twice the radius gives the number that the author was using as an approximation to π, which is $30/10 = 3$, as stated above.

Archimedes' dedication to geometry is suggested by the following quote.

'[Archimedes] … being perpetually charmed by his familiar siren, that is, by his geometry, he neglected to eat and drink and took no care of his person; that he was often carried by force to the baths.'

Attributed to Plutarch, AD 46–120.

The Greek mathematician Archimedes (in approximately 240 BC) devised a method for calculating π to any desired accuracy. The method involves sandwiching a circle of radius 1, which has a circumference of 2π, between two regular polygons, as shown in Figure 67.

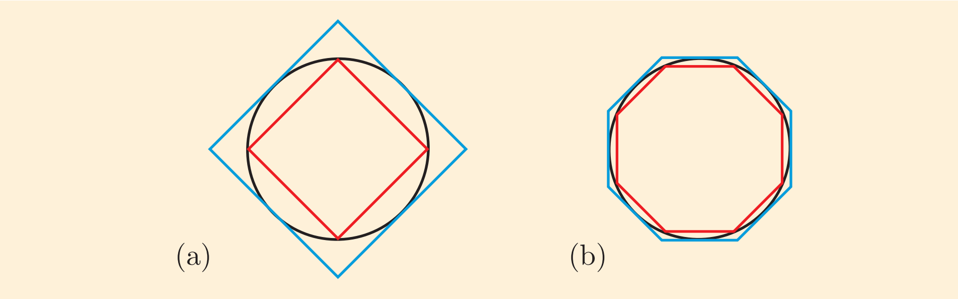


Figure 67 Approximating the circumference of a circle

Figure 67(a) shows a square **inscribed** in the circle, that is, drawn with corners touching the circle, and a larger square **circumscribed** outside the circle, that is, drawn with the centres of the sides touching the circle. Figure 67(b) shows the same circle, with inscribed and circumscribed octagons. The circumference of the circle is always sandwiched between the perimeters of the inscribed and circumscribed polygons. As the number of sides of the two polygons increases, the perimeters of the polygons provide closer approximations for the circumference of the circle. Since the exact circumference of the circle is 2π, dividing these approximations by 2 gives approximations for π. Archimedes used polygons with 96 sides to obtain $223/71 < \pi < 22/7$. This gives an approximation for π correct to two decimal places.

The fraction 22/7 was often used as an approximation to π before the widespread use of electronic calculators.

Another approximation by a fraction that is worth mentioning is 355/113, which was discovered by Chinese mathematician Zu Chongzhi in about AD 480. It is the closest approximation with the denominator below 1000, and is memorable because the sequence 113355 appears when you read it from bottom to top.

In the twentieth century, computer manufacturers used the calculation of π as a demonstration of the speed of their computers. As a result of this work, π has now been calculated to several billion decimal places. For practical purposes only a few decimal places are needed.

Most calculators have a button that gives an accurate approximation for π, and most mathematical software packages have a command that does this. In your calculations you should use these facilities rather than a simple approximation such as 22/7 or 3.14. For abstract problems it is usually best to avoid evaluating π at all and instead express answers in terms of π. For example, you might give the answers $\frac{3}{2}\pi$ or $\pi + 1$.

The following activity involves using the formula for the circumference of a circle.

Activity 32 *Estimating the length of the M25*

The M25 London orbital motorway is roughly circular with radius 25 km. Estimate the distance that a car travels as it circles London on the M25.

If you want to enclose a certain area of land with the shortest possible fence, then you should make your field circular. The circle is the shape that encloses the greatest possible area for a certain perimeter.

The constant π also appears in the formula for the area of a circle:

The area of a circle of radius r is πr^2.

This formula for the area of a circle has also been used to find approximations of π in the past, as the following activity shows.

Activity 33 *Estimating π from the Rhind papyrus*

The ancient Egyptian Rhind papyrus used the approximation that a circle of diameter 9 has the same area as a square of side 8. What value for π is given by this approximation?

The Rhind papyrus was mentioned in Unit 5.

The following example shows how the formula for the area of a circle can be used with other formulas to calculate the areas of more complicated shapes. This example also illustrates something that was mentioned in Unit 1 and which is often relevant to calculations involving areas of shapes: when you use intermediate results in later calculations, you should use the full-calculator-precision versions of the intermediate results, to avoid rounding errors.

For more examples on areas of circles, see Maths Help Module 7, Subsection 3.2.

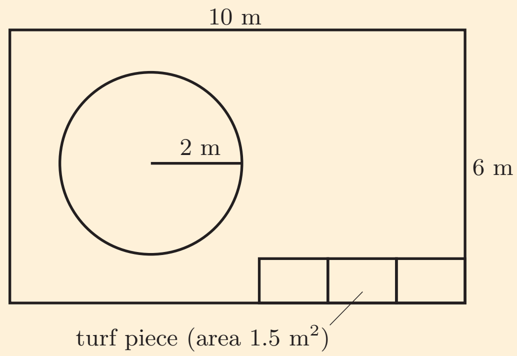
Example 14 *Finding areas*

A gardener wishes to turf a rectangular lawn and wants to estimate how much turf she should buy. The lawn is six metres by ten metres and has a circular flower bed of radius two metres cut into it. If the turf comes in pieces that cover $1.5\,\mathrm{m}^2$, how many pieces of turf should the gardener buy?

Solution

Collect the information in a diagram.

The essential information is usefully depicted as follows.



Calculate the relevant areas and combine them.

The area of the large rectangle is

$$6\,\mathrm{m} \times 10\,\mathrm{m} = 60\,\mathrm{m}^2.$$

The area of the circular flower bed is

$$\pi \times (2\,\mathrm{m})^2 = 12.56\ldots\ \mathrm{m}^2.$$

This gives the total area of lawn to be covered as

$$(60 - 12.56\ldots)\,\mathrm{m}^2 = 47.43\ldots\ \mathrm{m}^2.$$

Subtracting one area from another to calculate the area of a shape with holes in it is a useful general strategy.

So the number of pieces of turf required is

$$\frac{47.43\ldots}{1.5} = 31.62\ldots.$$

The gardener has to buy a whole number of pieces of turf, so she should buy 32 pieces and then cut some of these to fit around the circular flower bed.

Use the formula for the area of a circle to tackle the following problem.

Activity 34 *How much lawn seed?*

The gardener is now planning to seed a semicircular lawn of radius 3 m. The instructions on the box of seed say to scatter 50 g of seed for every square metre of lawn. How much lawn seed does the gardener need?

In this section, you have seen how to calculate the perimeters and areas of shapes composed of rectangles, triangles and circles. These types of calculations arise in many practical applications, such as estimating areas of land or calculating the quantities of materials needed for building or design projects.

5 Solids

Up until now, this unit has investigated plane shapes – shapes that can be drawn on a sheet of paper. Such shapes are said to be **two-dimensional** because they extend in two directions.

Geometry is not concerned only with two-dimensional shapes, but also with solid shapes such as spheres and cubes. These shapes, often known as **solids**, are said to be **three-dimensional** because they extend in three perpendicular directions. This short section introduces some three-dimensional shapes, and the related concepts of *volume* and *surface area*.

There are a number of words commonly used to describe the extent of a shape or an object in some direction; perhaps the five most familiar are length, breadth, width, height and depth. This section will primarily use the words width, height and depth, and will relate these to the way the shape or object is drawn in pictures as follows:

- *width* is the extent *across the page*
- *height* is the extent *up and down the page*
- *depth* is how far the object extends *into the page* (which is indicated by a perspective drawing).

5.1 Some standard solids

If you cut the solid in Figure 68 parallel to its ends, then the *cross-section* that you get always has the same shape and size as the ends. In other words, the cross-section is *uniform*. A solid with a uniform polygonal cross-section is called a **prism**. The prism in Figure 68 has a star-shaped cross-section, as shown.

A shape is *polygonal* if it is a polygon.

Another prism is shown in Figure 69: this one has a triangular cross-section, and a prism like this is called a **triangular prism**.

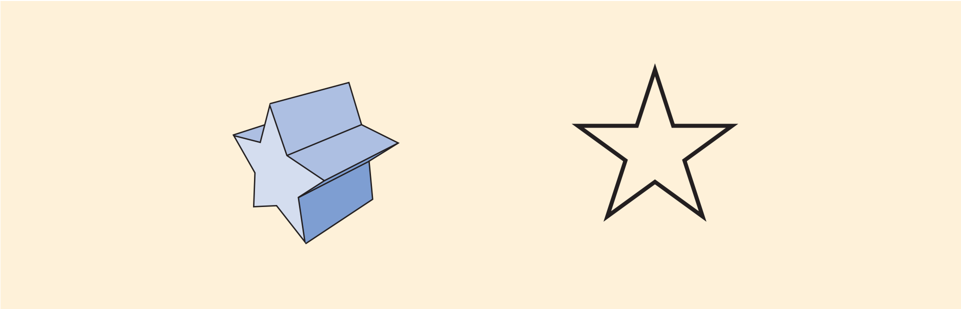


Figure 68 A star-shaped prism and a cross-section through it

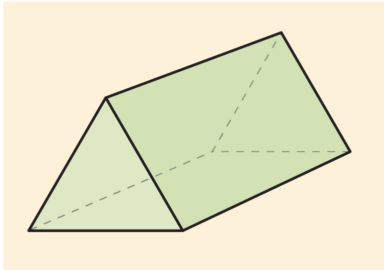


Figure 69 A triangular prism

A good way to think about a prism is to imagine filling in between two copies of a plane shape (the shape of the cross-section). In particular, a **cube** (Figure 70(a)) is an example of a prism, since it can be constructed by filling in between two copies of a square. (If the square has width and height a, then the two copies of the square must be distance a apart in order for the resulting solid to be a cube.) More generally, a **cuboid** (Figure 70(b)) – or more informally a box – is a prism whose cross-section is a rectangle. So a cube is a special case of a cuboid.

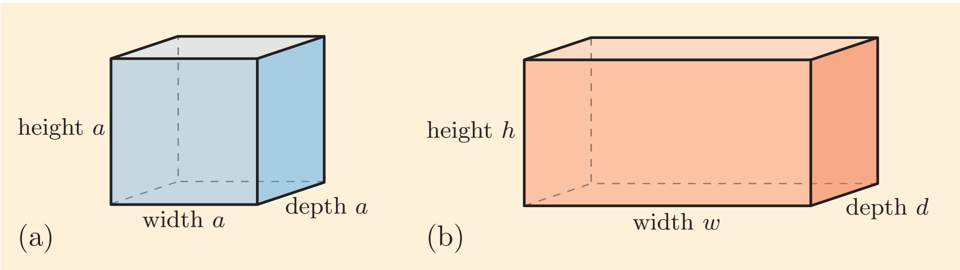


Figure 70 (a) A cube of width, height and depth a. (b) A cuboid of width w, height h and depth d.

Another important example of a solid with a uniform cross-section is a **cylinder** (Figure 71(a)), whose cross-section is a circle. (This solid isn't a prism because its cross-section isn't polygonal.) A cylinder is often drawn standing on one of its circular ends, so the distance between the two circles at its ends is usually denoted by h for height rather than d for depth.

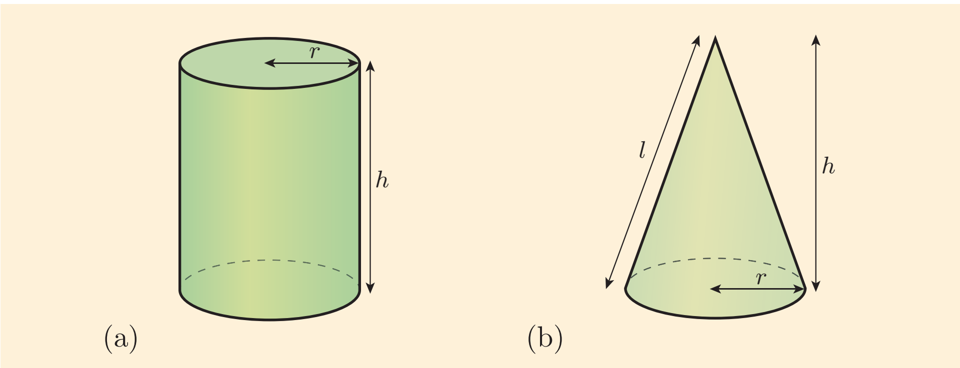


Figure 71 (a) A cylinder of radius r and height h. (b) A cone of height h whose base is a circle of radius r, with slant height l.

A related shape is a **cone** (Figure 71(b)), whose cross-sections are all circular but decrease in radius uniformly to a point, the **apex** of the cone. The **slant height** of a cone (denoted by l in Figure 71(b)) is the distance from the apex to any point on the circumference of the circular base.

In the prisms in Figures 68 and 69, each edge joining a vertex of one end of the prism to the matching vertex of the other end is perpendicular to the ends of the prism. Similarly, in the cylinder and cone in Figure 71, the line formed by the centres of the cross-sections is perpendicular to the base. In other texts you might see the words *prism*, *cylinder* and *cone* used to refer to solids in which these lines are not perpendicular to the base. Similarly, in other texts you might see the words *cylinder* and *cone* used to refer to solids with non-circular bases.

5.2 Volumes and surface areas of solids

Volumes

For more examples on volumes, see Maths Help Module 7, Subsection 3.4.

The **volume** of a solid object is the amount of physical space that the object occupies. Volume can be measured in *cubic centimetres* (cm^3), or *cubic metres* (m^3), and so on. A cubic centimetre is the volume of a cube whose edges all have length 1 cm (Figure 72), and a cubic metre is the volume of a cube whose edges all have length 1 m, and so on. A cubic centimetre is sometimes abbreviated as cc rather than cm^3; for example, you might hear about a 50 cc moped engine.

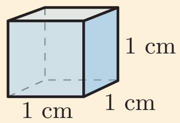


Figure 72 A cube with each edge of length 1 cm

A decimetre (dm) is one-tenth of a metre. Hence $1\,\text{dm} = 10\,\text{cm}$ and $1\,\text{dm}^3 = 1000\,\text{cm}^3$.

The concept of volume is closely related to the concept of **capacity**, which is the amount of liquid that an object could contain. Some units of capacity are litres in the metric system, and pints and gallons in British Imperial units. One litre is the capacity of one cubic decimetre (dm^3).

There are simple formulas for the volumes of many standard solid shapes, just as there are for the areas of standard plane shapes. For example, the volume of a cuboid of width w, height h and depth d is obtained by multiplying the three measurements together, so the formula for the volume is whd.

Another way to think about this formula for the volume of a cuboid is as the area of a cross-section, wd, times the height, h. This works for any solid with a uniform cross-section: any prism, and also (for example) a cylinder. You can think of such a solid as a thickening of one of its cross-sections, the thickness being the height h. Using this notion, you can see that the volume is obtained by multiplying the area of the cross-section (denoted by A in this unit) by the height, h. This gives the formula Ah for the volume.

This derivation can be made more formal and rigorous, as you will see if you study higher-level mathematics modules.

In the particular case of a cylinder, the cross-section is a circle. If the radius is r, then the area A is πr^2, so the volume of the cylinder is $Ah = \pi r^2 h$.

The formulas found above, together with formulas for the volumes of more complex shapes, are collected together for convenience in Table 4 on the next page.

Surface areas

Informally, the **surface area** of a solid can be thought of as the area of paper needed to wrap the object without any overlapping. The units of surface area are the same as the units of area, for example cm^2 or m^2.

The surface area of a box of width w, height h and depth d can be found by adding together the areas of the six rectangles that are the faces of the box. In order to visualise these six rectangles, imagine cutting along some of the edges of the box and unfolding the rest. The result is known as a **net** of the box (see Figure 73).

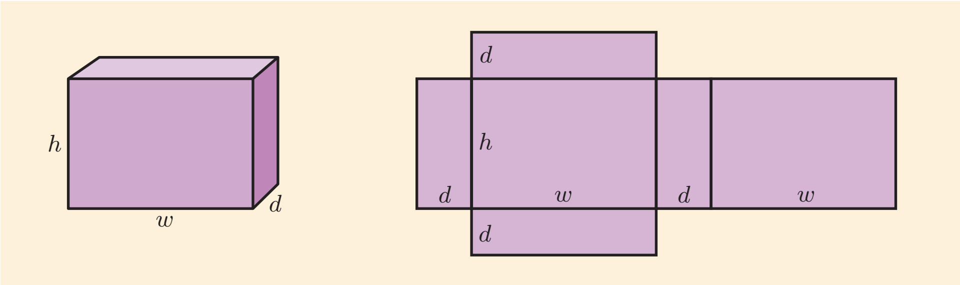
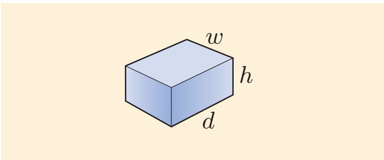
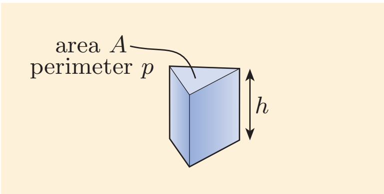
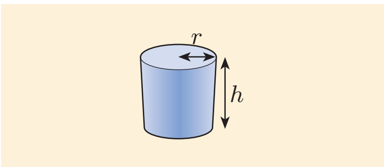
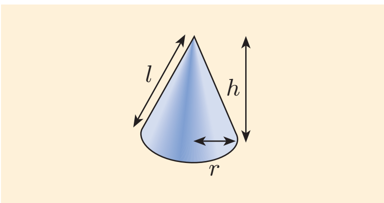
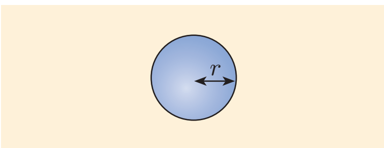


Figure 73 A box and a net of the box

Adding together the areas of the six faces gives the formula for the surface area as $2wh + 2wd + 2hd$.

Formulas for the surface areas of more complex shapes are given in Table 4, which collects together formulas for volumes and surface areas.

Table 4 Volumes and surface areas of simple solids

Shape	Diagram	Volume	Surface area
Cuboid		whd	$2wh + 2wd + 2hd$
Prism		Ah	$2A + hp$
Cylinder		$\pi r^2 h$	$2\pi r^2 + 2\pi rh$
Cone		$\frac{1}{3}\pi r^2 h$	$\pi r^2 + \pi rl$
Sphere		$\frac{4}{3}\pi r^3$	$4\pi r^2$

There is something worth noticing about the entries in Table 4. In each formula for surface area, each term is of the form

constant × length × length   or   constant × area.

This is because in each case the unit needs to be a unit of area, such as m^2.

The sphere is the shape with minimum surface area for a certain volume. This is why soap bubbles in the air are spherical: the volume of air inside the bubble is fixed, and surface tension in the soap film acts to make the film have the smallest possible area. (You saw a similar result about the areas and circumferences of circles earlier.)

Similarly, in each formula for volume, each term is of the form

$$\text{constant} \times \text{length} \times \text{length} \times \text{length} \quad \text{or} \quad \text{constant} \times \text{area} \times \text{length}.$$

This is because in each case the unit needs to be a unit of volume, such as m^3. You can use these facts as a quick check on whether you have a correct formula. For example, if you are calculating a volume, then the formula should involve multiplying three lengths, or an area and a length, together.

5.3 Using volumes and surface areas

The following example shows how to apply one of the formulas in Table 4.

If you want to make a bucket with a certain volume, but using the smallest area of sheet material, then you should make the diameter of the bucket equal to twice its height. Real buckets are not this optimal shape, probably because such a wide bucket would be hard to carry.

On the other hand, if you want a bucket with a lid, you can enclose the greatest volume with the smallest amount of material if you make the height and diameter of the cylinder equal. It's probably no coincidence that paint tins often have roughly these proportions.

Example 15 *Finding the capacities of buckets*

A window cleaner has two cylindrical buckets, one with diameter 25 cm and height 30 cm, and the other with diameter 30 cm and height 25 cm. Which will hold more water?

Solution

The capacities of the buckets can be calculated by using the formula for the volume of a cylinder, $\pi r^2 h$. For the first bucket the height is $h = 30$ cm and the radius is $r = 25/2 = 12.5$ cm, which gives the volume in cm^3 as

$$\pi \times 12.5^2 \times 30 \approx 14\,726.$$

Since $1000\,cm^3$ is a litre, the capacity of the bucket is about 15 litres.

The second bucket has height $h = 25$ cm and radius $r = 30/2 = 15$ cm, so its volume in cm^3 is

$$\pi \times 15^2 \times 25 \approx 17\,671,$$

which represents a capacity of about 18 litres.

The capacity of the second bucket is larger, so it holds more water.

Activity 35 *Finding the surface area of a metal can*

What is the surface area of a metal can that is a cylinder of diameter 7.5 cm and height 10.5 cm?

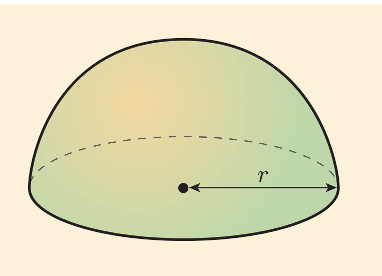


Figure 74 A hemisphere of radius r

Here are two final activities for you to try. They involve a solid called a *hemisphere*, which means half a sphere, as shown in Figure 74. More precisely, a **hemisphere** is the solid shape obtained by cutting a sphere along a plane through its centre.

Figure 75 Greenhouses at the Eden Project in Cornwall

Activity 36 *How much foil?*

A dome is to be constructed out of steel and transparent foil in a similar way to the greenhouses shown in Figure 75. Model the dome as a hemisphere of radius 10 m to calculate the area of foil required to construct the dome to the nearest ten square metres.

Activity 37 *How many ice cream cones?*

This question concerns ice cream cones of diameter 6 cm and height 12 cm, filled with ice cream all the way to the bottom of the cone and heaped on top in such a way that the upper surface is a hemisphere.

(a) Calculate the volume of ice cream needed to make an ice cream cone to the above description. Give your answer in cubic centimetres rounded to the nearest cubic centimetre.

(b) Convert the volume of ice cream found in part (a) to litres.

(c) Hence calculate the number of ice cream cones of the above description that could be made from a one-litre tub of ice cream.

Many modern buildings use geometric properties of solid shapes in innovative ways. One of the most famous is the Sydney Opera House, shown in Figure 76. The curved surfaces are parts of spheres, all with the same radius.

Figure 76 Sydney Opera House

This unit has introduced the basics of geometry. In Unit 12 you will see how to calculate the areas of more complicated shapes, but there are many other important and useful areas of geometry, which you may encounter if you continue with further study in mathematics. For example, you could learn about the perspective that artists use to represent three-dimensional objects on a two-dimensional canvas (this is called *projective geometry*), or you could study the geometry of regular patterns such as tiling patterns, and their rotational and mirror symmetries, which are considerably more varied than the symmetries that you have seen in this unit.

Learning checklist

After studying this unit, you should be able to:

- determine angles in shapes using the following facts:
  - the angles in a triangle add up to 180°
  - the angles on a straight line add up to 180°
  - the angles making up a right angle add up to 90°
  - when two lines cross, the opposite (X) angles are equal
  - when a line crosses two parallel lines, the corresponding (F) angles are equal
  - when a line crosses two parallel lines, the alternate (Z) angles are equal
- construct geometric arguments
- identify line and rotational symmetries
- show that two triangles are congruent
- calculate the lengths of corresponding sides in similar triangles
- use Pythagoras' Theorem to calculate the length of a side of a right-angled triangle when given the other two lengths
- calculate the areas and perimeters of some shapes constructed from rectangles, triangles and parts of circles
- calculate the volumes and surface areas of simple solids.

Solutions and comments on Activities

Activity 2

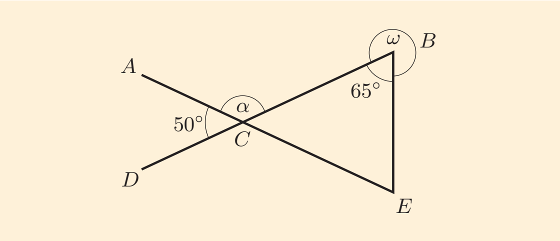
(a) BCD is a straight line. So

$\alpha + 50° = 180°$ (angles on a straight line)

$\alpha = 180° - 50° = 130°$.

(b) $\angle ACB$ and $\angle DCE$ are obtuse.

(c)



The angles in a full turn add up to 360°. So

$\omega = 360° - 65° = 295°$.

Activity 3

Angles on a straight line add up to 180°. So

$2\theta + 110° = 180°$

$2\theta = 70°$

$\theta = 35°$.

Activity 4

(a) The marked angles are always equal.

(b) The alternate angles α and γ are equal. See the comments in the text panel in the dynamic geometry resource.

Activity 5

(a) $\theta + \psi = 180°$ (angles on a straight line).

(b) $\psi + \phi = 180°$ (angles on a straight line).

(c) Subtracting the equation in part (b) from the equation in part (a) gives

$\theta - \phi = 0°$.

Hence $\theta = \phi$.

(You might have eliminated ψ in a different way.)

Activity 6

Since $\angle ABC$ and $\angle HBE$ are opposite angles, $\angle HBE = 40°$.

Also, $\angle FAD$ is opposite to $\angle BAC$, so $\angle FAD = 70°$.

Finally, FAC is a straight line, so $\angle BAF = 180° - 70° = 110°$.

Activity 7

$\angle BEF$ is opposite $\angle DEG$ and is therefore also equal to 50°.

$\angle ABE$ is an alternate angle with $\angle BEF$ (and also a corresponding angle with $\angle DEG$), and so is equal to 50°.

Finally, $\angle HBC$ is opposite $\angle ABE$ (and also a corresponding angle with $\angle BEF$), and so is also equal to 50°.

Activity 8

The three angles marked θ are angles on a straight line and hence add to 180°. Thus

$3\theta = 180°$

$\theta = 60°$.

So

$\angle ABE = 2 \times 60° = 120°$.

$\angle ABE$ and the angle marked ϕ are alternate angles, so

$\phi = 120°$.

Activity 9

Follow the instructions in the text panel of the 'Triangle' tab in the dynamic geometry resource to see how the result about the angles in a triangle can be proved.

Activity 10

The three angles of a triangle add up to 180°. In an equilateral triangle these angles are also equal, so each of the angles is $\frac{180}{3} = 60°$.

Activity 11

To find the required angles, use the fact that the angles of a triangle add up to 180°.

(a) The two base angles are equal to 62°. Let θ be the apex angle. Then

$\theta + 62° + 62° = 180°$

$\theta = 180° - 124°$

$\theta = 56°$.

So the apex angle is 56°.

(b) Let θ be the size of each base angle of the isosceles triangle with apex angle 30°. Then

$\theta + \theta + 30° = 180°$

$2\theta = 150°$

$\theta = 75°$.

So the base angles are each 75°.

Activity 12

Each of the matching wooden strips makes an angle of 65° with the vertical. Thus $\angle DKH = 65°$.

Since $\angle DKH$ and $\angle HKC$ are angles on a straight line, they add to 180°. Therefore

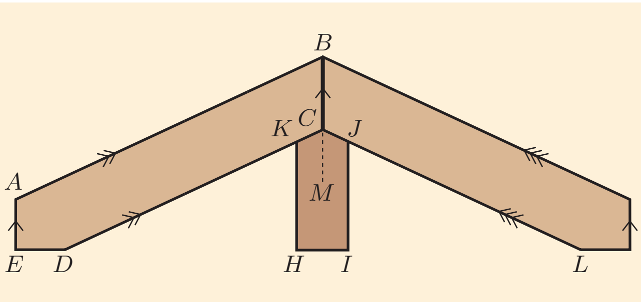
$$\angle HKC = 180° - 65° = 115°.$$

Similarly, $\angle LJI = 65°$, and so $\angle CJI = 115°$.

There are at least two ways to find $\angle KCJ$. One way is to note that $\angle HKC$ and $\angle KCB$ are alternate angles, as are $\angle IJC$ and $\angle JCB$. Thus both $\angle KCB$ and $\angle JCB$ are equal to 115°. Therefore

$$\angle KCJ = 360° - 2 \times 115° = 130°.$$

Another way is to extend the line BC, say to M.



Now, since the matching strips each make an angle of 65° with the vertical, it follows that $\angle KCM = \angle MCJ = 65°$.

Since $\angle KCJ = \angle KCM + \angle MCJ$, it follows that

$$\angle KCJ = 2 \times 65° = 130°.$$

Activity 13

(a) Using the fact that the angles in a triangle add up to 180° gives

$$\angle CBD = 180° - 30° - \theta = 150° - \theta.$$

But AD is a straight line, so $\angle ABC + \angle CBD = 180°$, which gives

$$\begin{aligned} \angle ABC &= 180° - \angle CBD \\ &= 180° - (150° - \theta) \\ &= 180° - 150° + \theta \\ &= 30° + \theta. \end{aligned}$$

(b) Using the fact that the angles in a triangle add up to 180° gives $\angle ABD = 180° - 90° - \theta$, which simplifies to

$$\angle ABD = 90° - \theta.$$

Now $\angle ABC$, $\angle ABD$ and $\angle DBE$ add to 180° as they are angles on the straight line EC. Since $\angle DBE$ is 90°, and we have just found that $\angle ABD = 90° - \theta$, this gives the equation

$$\angle ABC + (90° - \theta) + 90° = 180°.$$

So

$$\begin{aligned} \angle ABC - \theta &= 0° \\ \angle ABC &= \theta. \end{aligned}$$

Activity 14

The conjecture is that the sum of the exterior angles of a triangle add up to 360°.

Each exterior angle of the triangle is on a straight line with the corresponding interior angle of the triangle. Therefore if the exterior angles are α, β and γ, then the interior angles are $180° - \alpha$, $180° - \beta$ and $180° - \gamma$.

From the result that the interior angles of a triangle add up to 180°, it follows that

$$\begin{aligned} (180° - \alpha) + (180° - \beta) + (180° - \gamma) &= 180° \\ 540° - (\alpha + \beta + \gamma) &= 180° \\ 540° - 180° &= \alpha + \beta + \gamma \\ \alpha + \beta + \gamma &= 360°. \end{aligned}$$

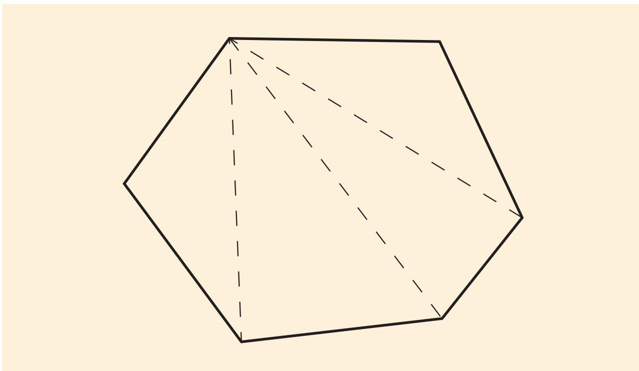
This proves the conjecture.

Activity 15

Read the discussion in the 'Parallelogram' tab of the dynamic geometry resource – this demonstrates that opposite angles of a parallelogram are equal.

Activity 16

A hexagon divided into triangles is shown below.



From the figure you can see that the hexagon has been split into four triangles, and each angle in each triangle is an angle in the hexagon or a part of one of these angles. So the angle sum of a hexagon is $4 \times 180° = 720°$.

(You may have split up your hexagon differently; there are several ways to do it, but they all give four triangles.)

Activity 17

The solution to Activity 16 shows that the angle sum of a hexagon is 720°. Since there are six equal interior angles in a regular hexagon, each angle is

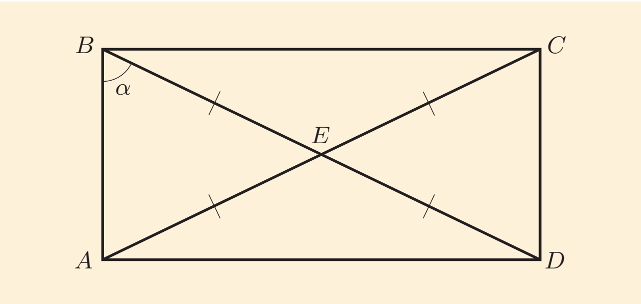
$$\frac{720°}{6} = 120°.$$

Activity 18

(a) Since AE and BE represent equal lengths of rope, $\triangle ABE$ is an isosceles triangle. So its base angles, $\angle ABE$ and $\angle BAE$, are equal.

Similarly, $\triangle BCE$ is an isosceles triangle. So its base angles, $\angle EBC$ and $\angle ECB$, are equal.

(b) In the diagram below, $\angle ABE$ is labelled as α.



(i) $\angle BAE = \alpha$.

(ii) Since the interior angles of a triangle add up to 180°,

$$\angle AEB = 180° - \alpha - \alpha = 180° - 2\alpha.$$

(iii) Since angles on a straight line add up to 180°,

$$\begin{aligned}\angle BEC &= 180° - \angle AEB\\ &= 180° - (180° - 2\alpha)\\ &= 180° - 180° + 2\alpha\\ &= 2\alpha.\end{aligned}$$

(iv) Since the interior angles of a triangle add up to 180°,

$$\angle EBC + \angle ECB + \angle BEC = 180°;$$

that is,

$$\angle EBC + \angle ECB + 2\alpha = 180°.$$

So

$$\angle EBC + \angle ECB = 180° - 2\alpha.$$

Since $\angle EBC = \angle ECB$ (from part (a)), it follows that

$$2 \times \angle EBC = 180° - 2\alpha,$$

so

$$\angle EBC = 90° - \alpha.$$

(c) Hence

$$\angle ABE + \angle EBC = \alpha + (90° - \alpha) = 90°.$$

So $\angle ABC$, which is $\angle ABE + \angle EBC$, is a right angle.

(d) Since all the angles are 90°, $ABCD$ is a rectangle.

Activity 19

(a) This is a regular octagon, so it has rotational symmetry of order 8 (the number of sides).

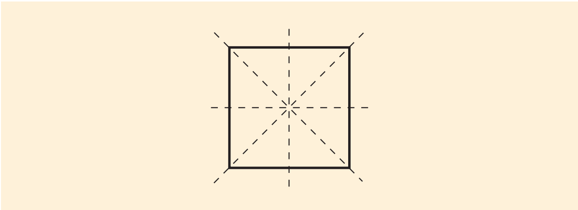
(b) The letter 'S' has rotational symmetry of order 2.

(c) This picture can be rotated 8 times around the centre before returning to its starting point and each time the picture will look the same, so the order of rotational symmetry is 8.

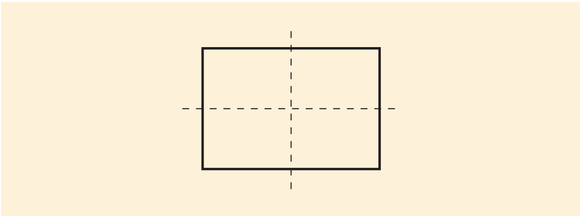
(d) Each of the five depressed sections in the wheel can be rotated to the next, so the order of rotational symmetry is 5. (This ignores the faint logo at the centre of the wheel.)

Activity 20

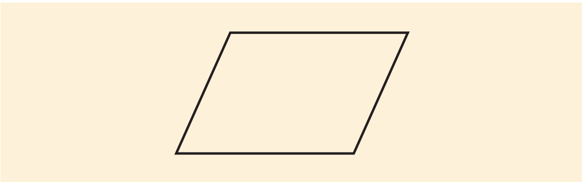
(a) A square has four lines of symmetry.



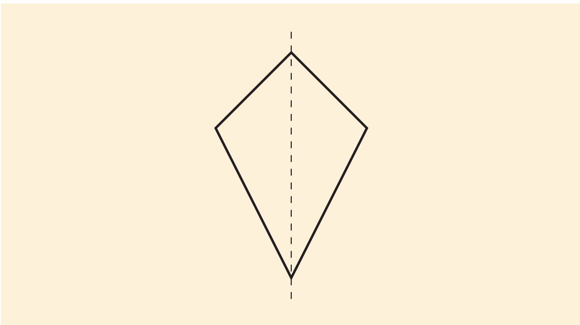
(b) A rectangle has two lines of symmetry.



(c) A parallelogram has no lines of symmetry. (It does, however, have rotational symmetry of order 2.)



(d) A kite has just one line of symmetry.



Activity 21

(a) Each triangle has sides of lengths 3, 4 and 6. So the two triangles are congruent by SSS.

(b) In this case we are given two pairs of sides equal and the included angles equal, so the two triangles are congruent by SAS.

(c) The three angles are the same in both triangles. However, although each triangle has a side of length 2, these sides are not corresponding. So the triangles are different sizes and hence are not congruent.

(d) Two angles in one triangle are equal to two angles in the other triangle, so the remaining angles in the triangles are also equal. However, the two triangles may be different sizes, so they are not necessarily congruent.

Activity 22

First notice that $\angle DCE = 50°$, since it is opposite to $\angle ACB$ at vertex C.

In $\triangle ABC$ and $\triangle DEC$:

- $AC = DC$
- $\angle ACB = \angle DCE$ (both angles are $50°$)
- $BC = EC$.

So $\triangle ABC \cong \triangle DEC$ (by SAS).

Activity 23

(a) First notice that $\angle DCE = \theta$ since it is opposite to $\angle ACF$ at vertex C.

In $\triangle BCE$ and $\triangle DCE$:

- $\angle BCE = \angle DCE$ (both angles are θ)
- the side CE is common to both triangles
- $\angle BEC = \angle DEC$ (both angles are $90°$).

So $\triangle BCE \cong \triangle DCE$ (by ASA).

(b) The line segments BE and DE are corresponding sides in congruent triangles (since they are both opposite to the angle θ), so they are equal.

Activity 24

$\triangle LMN$, $\triangle RST$ and $\triangle XYZ$ are similar because these triangles have the same three angles as each other. So the ratios of corresponding sides are equal. First consider $\triangle LMN$ and $\triangle RST$. The sides MN and ST are corresponding since they are both opposite the angle marked with one arc. Similarly, NL and TR are corresponding, and ML and SR are also corresponding. So

$$\frac{ST}{MN} = \frac{TR}{NL} = \frac{SR}{ML}.$$

Substituting in the known lengths gives

$$\frac{ST}{5} = \frac{10}{8} = \frac{SR}{10}.$$

So

$$ST = \frac{5 \times 10}{8} = \frac{50}{8} = 6.25$$

and

$$SR = \frac{10 \times 10}{8} = \frac{100}{8} = 12.5.$$

Now consider $\triangle XYZ$ and $\triangle LMN$. This gives

$$\frac{XZ}{8} = \frac{20}{5} = \frac{YX}{10}.$$

So

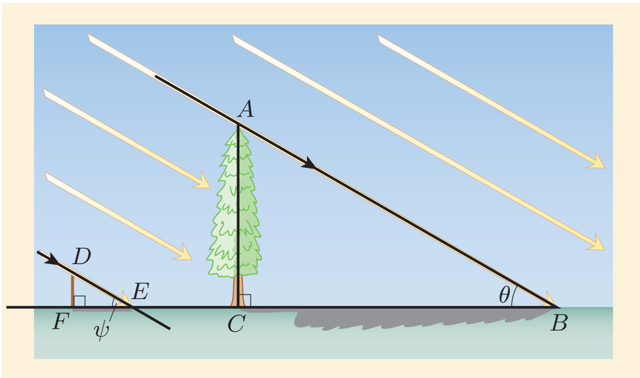
$$XZ = \frac{20 \times 8}{5} = 32,$$

and

$$YX = \frac{20 \times 10}{5} = 40.$$

Activity 25

(a) The Sun's rays are parallel, so we have the following diagram.



The angles θ and ψ are equal because they are a pair of corresponding (F) angles. Also, each of the two triangles contains a right angle. So two angles in one triangle are equal to two angles in the other triangle. Therefore $\triangle ABC$ is similar to $\triangle DEF$.

(b) Since $\triangle ABC$ is similar to $\triangle DEF$, the ratios of corresponding sides are equal, so

$$\frac{AC}{DF} = \frac{BC}{EF}.$$

Now $BC = 32.5$, $EF = 1.6$ and $DF = 1$, where all measurements are in metres.

Hence

$$\frac{AC}{1} = \frac{32.5}{1.6},$$

so

$$AC = \frac{32.5}{1.6} \approx 20.3.$$

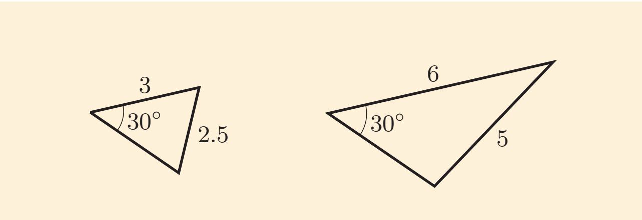
Hence the height of the tree is approximately 20.3 metres.

Activity 26

(a) These two triangles are not similar. The ratio of the shortest side of the second triangle to the shortest side of the first is $2:1$, since $4 = 2 \times 2$, so if these triangles were similar, the scale factor would be $k = 2$. But the other two pairs of sides are not in the ratio $2:1$, so they don't correspond to $k = 2$.

(b) Since the angles in a triangle add up to $180°$, each triangle has angles of $100°$, $50°$ and $30°$. So the angles are the same and hence the triangles are similar.

(c) Here two pairs of sides have the same ratio, namely $k = 2$, since $5 = 2 \times 2.5$ and $6 = 2 \times 3$. However, the marked angle is not included, and it is possible to draw two triangles with these dimensions that are not similar. For example,



(d) Here two pairs of sides have the same ratio and there is an included angle, so the triangles are similar.

Activity 27

(a) By Pythagoras' Theorem,

$$RQ^2 = PR^2 + PQ^2 = 3^2 + 4^2 = 25.$$

So

$$RQ = \sqrt{25} = 5.$$

(The negative square root is disregarded as lengths are positive.)

(b) Let the length of the unknown side be c. By Pythagoras' Theorem,

$$c^2 = 2^2 + 5^2 = 29.$$

So $c = \sqrt{29}$.

(As the problem is abstract, it is not necessary to calculate a decimal approximation such as $\sqrt{29} \approx 5.39$.)

(c) Let the length of the unknown side be b. By Pythagoras' Theorem,

$$3^2 + b^2 = 10^2,$$

that is,

$$9 + b^2 = 100,$$

thus

$$b^2 = 91.$$

So $b = \sqrt{91}$.

(d) By Pythagoras' Theorem,

$$EF^2 = ED^2 + DF^2.$$

Substituting $DF = 1.2$ and $EF = 5$ gives

$$5^2 = ED^2 + 1.2^2.$$

So

$$ED^2 = 5^2 - 1.2^2 = 25 - 1.44 = 23.56.$$

Hence

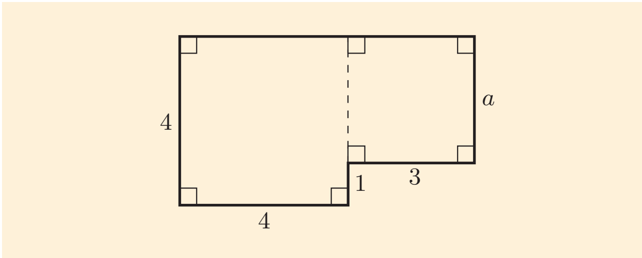
$$ED = \sqrt{23.56} \approx 4.85.$$

That is, the length of the third side is approximately 4.85 m.

(Here the answer is calculated as a decimal value, since the problem is a practical one involving lengths with units.)

Activity 29

(a) The shape can be split into two rectangles in different ways. For example, consider the following split.



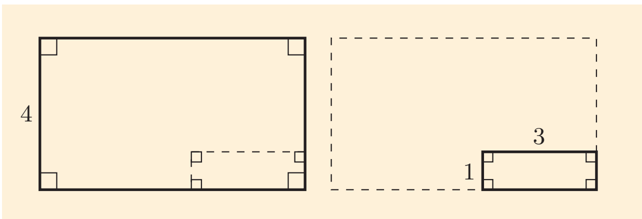
The next step is to calculate the length a in the diagram. This can be done by looking at the lengths of the other vertical sides, which gives

$$a = 4 - 1 = 3.$$

So the area of the shape is

$$4 \times 4 + 3 \times 3 = 16 + 9 = 25.$$

(An alternative way to calculate the area is to think of the shape as the difference between two rectangles, as follows.



The width of the larger rectangle is $4+3=7$, so the area of the shape is

$$4\times 7 - 1\times 3 = 28 - 3 = 25,$$

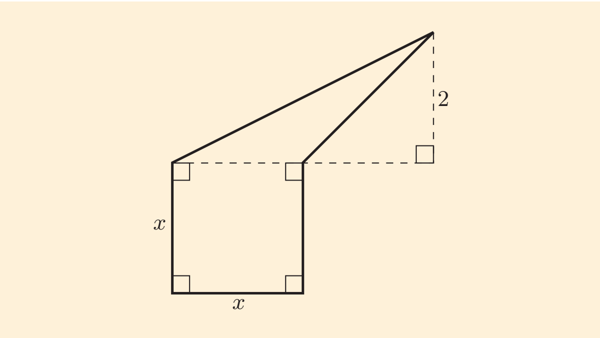
as found previously.)

(b) The area of this triangle can be calculated directly by using the formula $\frac{1}{2}bh$. The base of the triangle can be taken to be the uppermost side. Then the base, b, is 3 and the perpendicular height, h, is 2, so the area of the triangle is

$$\tfrac{1}{2}bh = \tfrac{1}{2}\times 3\times 2 = 3.$$

(c) Using the formula for the area of a parallelogram of base 5 and perpendicular height 3 gives the area as $5\times 3 = 15$.

(d)



This shape splits up into a square with sides x and a triangle with base x and perpendicular height 2. The area of a square with sides x is $x\times x = x^2$. The area of a triangle with base x and perpendicular height 2 is $\frac{1}{2}\times x\times 2 = x$. So the area of the shape is x^2+x.

(The expression x^2+x is an example of a *quadratic expression*, so this shows how such expressions arise naturally when calculating the areas of shapes with sides of unknown length. You will learn about quadratic expressions in Units 9 and 10.)

Activity 30

The answers are in the table below.

	Area	Perimeter	Wiggliness
(a)	5	12	$\frac{144}{5} = 28.8$
(b)	20	24	$\frac{24^2}{20} = 28.8$
(c)	45	36	$\frac{36^2}{45} = 28.8$

(You can see that the wiggliness of the shape is unchanged by enlargements with scale factor 2 or 3. This outcome is discussed in the text following the activity.)

Activity 31

(a) Your completed table should be as follows.

Cell	Area	Perimeter	Wiggliness
Small normal	83	42	21.3
Abnormal	83	62	46.3
Large normal	150	62	25.6

(b) Based on the table, a threshold of between 30 and 40, say, might be appropriate to distinguish the two types of cells. For example, the criterion might be that cells with a wiggliness less than 35 are classified as normal, while those with a wiggliness greater than 35 are classed as abnormal.

(Much more data would need to be collected before a more precise threshold could be defined. A trial that compared the calculated wiggliness of cells against human judgement of abnormality could provide these data.)

Activity 32

The radius of the motorway is 25 km. So the circumference is

$$2\times\pi\times 25\,\text{km} \approx 157.080\,\text{km}.$$

It would be ridiculous to quote the answer to this precision, since the motorway is only approximately circular and the given radius, 25 km, is also approximate. Rounding to the nearest 10 km seems sensible, so an appropriate answer is that a car travels about 160 km.

(This rough calculation compares moderately well with the actual length of the M25, which is 188.3 km.)

Activity 33

A square of side 8 has area $8\times 8 = 64$. A circle of diameter 9 has radius 4.5 and hence area $\pi\times 4.5^2$. Assuming that the square and circle have the same area gives the equation

$$64 = \pi\times 4.5^2.$$

Making π the subject of this equation gives

$$\pi = \frac{64}{4.5^2} \approx 3.16.$$

So the approximation for π implicit in this part of the Rhind papyrus is

$$\pi \approx 3.16.$$

(This approximation is quite close to the true value; it is accurate to within 1%.)

Activity 34

The area of a semicircle of radius r is $\frac{1}{2}\pi r^2$. If the radius is 3 m, then the area in m^2 is

$$\frac{1}{2} \times \pi \times 3^2 = \frac{9\pi}{2}.$$

So the amount of grass seed needed in grams is

$$\frac{9\pi}{2} \times 50 = 700 \text{ (to 1 s.f.)}.$$

Hence approximately 0.7 kg of seed is needed.

Activity 35

The surface area A of a cylinder of radius r and height h is given by the formula

$$A = 2\pi r^2 + 2\pi rh.$$

The diameter of the can is 7.5 cm, so the radius is 3.75 cm.

Substituting $r = 3.75$ and $h = 10.5$ into the formula gives

$$\begin{aligned} A &= 2 \times \pi \times (3.75)^2 + 2 \times \pi \times 3.75 \times 10.5 \\ &\approx 335.75\ldots. \end{aligned}$$

So the surface area of the can is approximately 340 cm^2.

Activity 36

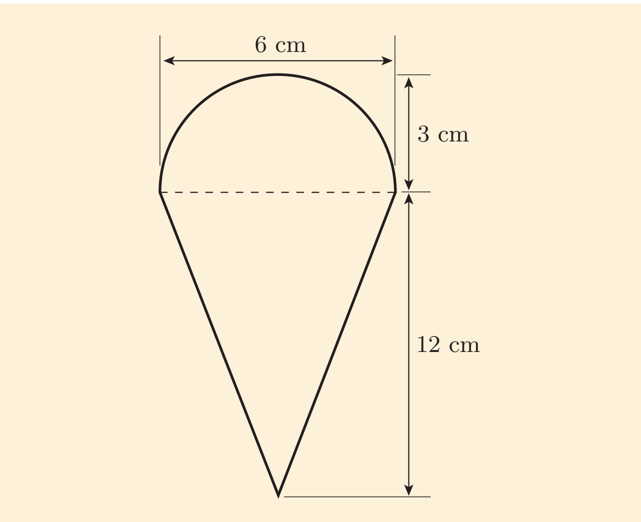
The area of foil required is half of the surface area of a sphere of radius 10 m. The surface area of a sphere of radius r is given by the formula $4\pi r^2$, so the area, in m^2, of foil required for the dome is

$$\tfrac{1}{2} \times 4 \times \pi \times 10^2 \approx 628.31\ldots.$$

So, to the nearest ten square metres, the dome will require 630 m^2 of foil.

Activity 37

(a) The first step is to draw a diagram and add all relevant information. For this question you don't need to draw a three-dimensional shape: you can write all the relevant information on a diagram of a cross-section of the shape, as shown below.



This shape splits into upper and lower parts as indicated by the dashed line in the diagram.

The lower part of the shape is a cone, so its volume is given by the formula $\frac{1}{3}\pi r^2 h$. The radius r is 3 cm and the height h is 12 cm, so the volume in cm^3 is

$$\tfrac{1}{3} \times \pi \times 3^2 \times 12 = 113.09\ldots.$$

The upper part of the shape is half of a sphere. The volume of a sphere of radius r is given by the formula $\frac{4}{3}\pi r^3$, so the volume, in cm^3, of the upper part is

$$\tfrac{1}{2} \times \tfrac{4}{3} \times \pi \times 3^3 = 56.54\ldots.$$

So the total volume in cm^3 of the ice cream is

$$113.09\ldots + 56.54\ldots = 169.64\ldots.$$

To the nearest cubic centimetre the volume is 170 cm^3.

(b) There are 1000 cm^3 in one litre. So the number of litres of ice cream in one cone is

$$\frac{169.64\ldots}{1000} = 0.169\ldots,$$

that is, each ice cream cone is made from approximately 0.17 litres of ice cream.

(c) The number of ice cream cones that can be made can be found by dividing the amount of ice cream in the tub by the amount of ice cream required to make one cone:

$$\text{number of cones} = \frac{1}{0.169\ldots} = 5.894\ldots.$$

So the one-litre tub of ice cream will make 5 full ice cream cones (with enough left over to make 89% of another cone).